THERMOPHYSICS '99

Meeting of the Thermophysical Society
Working Group of the Slovak Physical Society

Bratislava, October 22, 1999

Editor Libor Vozár

Constantine the Philosopher University in Nitra
Faculty of Natural Sciences
1999

THERMOPHYSICS ‘99
Proceedings of the Meeting of the Thermophysical Society - Working Group of the Slovak Physical Society
Bratislava, October 22, 1999

Editor Libor Vozár

Issued by Constantine the Philosopher University in Nitra
First Edition
Released in 1999
Printed by Car Print Nitra

Published with the support by the Slovak Science Grant Agency under the contract 1/6115/99.

ISBN 80-8050-284-6

CONTENTS

PERFACE

Dear colleagues,

Just four years ago our activity due to the Workshop on Measuring the Thermophysical Properties has started. Prof. Barta organized the first meeting on January 22, 1996 at the Department of Physics, Faculty of Electrical Engineering and Information Technology at the Slovak Technical University in Bratislava. Participants coming from universities and research institutes have given information regarding contemporary state in measuring methods and specimen geometry that have used in their laboratories. One of the conclusions of this workshop was the recommendation regarding regular meetings that should be organized every year.

Dr. Kubičár from the Institute of Physics organized the second workshop on June 13, 1997. The researchers from the Departments of Physics of the Slovak Technical University in Bratislava and the Department of Physics of the Constantine the Philosopher University in Nitra, Institute of Metrology, Institute of Construction and Architecture, Institute of Measurement Science and Institute of Physics of the Slovak Academy of Sciences in Bratislava participated in this workshop. Several interesting contributions were presented and were devoted to the theory of measurements, experimental technique as well as to the investigations of the heat transport in inhomogenous materials. A decision was done that the Thermophysical Society – society that includes researchers working in thermophysics should be created. Thermophysical research has had a long tradition and good reputation in Slovakia especially due to the pioneer works of Prof. Krempaský acting at the Department of Physics, Faculty of Electrical Engineering and Information Technology of the Slovak Technical University in Bratislava. Prof. Krempaský together with his scientific school has written several fundamental works that has been devoted to rapidly developing area of transient methods.

Thermophysical Society should bring together researchers that are active or should be active in future in the following areas:

- study of the heat transport,
- investigation of thermophysical properties of materials,
- development of the theory of measurement methods,
- development of experimental techniques for measuring the thermophysical properties,
- application of thermophysics in technology, etc.

An annual workshop should be an occasion where the members present contributions and share experiences regarding to their activity in research. Web page was constructed in which all activity of Thermophysical Society is consecutively recorded. Thermophysical Society has been accepted as a regular Working Group within the framework of the Slovak Physical Society.

Activity of the Thermophysical Society continued again in 1998 due to the workshop held on October 2 1998 at Institute of Physics. Prof. Černý and Prof. Toman from the Department of Physics, Faculty of Civil Engineering at the Czech Technical University in Prague participated in this workshop, too. In addition, Dr. Gustafsson from the University of Gothenburg visited Institute of Physics on November 20 1998 and he gave a talk “Measuring Technique by Hot Disc Method”. Later in 1999 Prof. Assael from the Faculty of Chemical Engineering of the Aristotle University of Thessaloniki visited

Constantine the Philosopher University in Nitra and on April 7 he gave a talk "Transport Properties of Fluids: Research Activities in Greece".

In 1999 the Workshop was held on October 22 at the Institute of Physics. Traditionally, researcher from the Slovak Technical University in Bratislava, Constantine the Philosopher University in Nitra, Institute of Construction and Architecture, Institute of Measurement Science, Institute of Physics and Institute of Metrology of the Slovak Academy of Sciences in Bratislava participated on the program by interesting contributions and discussion. In contrary to previous meetings a proceedings is released where the contributions are printed. The aim of the proceedings is to present activities of the working group. Similar proceedings will be published every year.

Next year, the workshop will be held in Nitra. Prof. Vozár should continue the organization of traditional workshops. We wish him a good success in the organizing of the following meetings.

I was acting as the chairman of Thermophysical Society in the years 1997 – 1999. Prof. Vozár was appointed to be a chairman for the next period 2000 – 2003. Let's wish him good start in his activities within the framework of Thermophysical Society.

Ľudovít Kubičár

EFFECTIVE HEAT EQUATION IN PARTICULATE COMPOSITE

Štefan Barta

Department of Physics, Faculty of Electrical Engineering and Information Technology,
Slovak Technical University, Ilkovičova 3, SK-812 19 Bratislava, Slovakia
Email: bartas@elf.stuba.sk

Abstract

The effective heat equation and the formula for effective thermal conductivity in a particulate composite are derived in this paper.

Key words: effective heat equation, effective thermal conductivity, particulate composite

1 Introduction

The aim of this paper is to derive the heat equation, which describes the heat conduction in composite materials. The composite materials are regarded as those ones, consisting of grains of the various components possibly located in matrix. For the understanding and interpretation of the properties of composite materials one has to notice the structure of grains on the submacroscopic level, i.e. on the level of the linear dimension of grains. At this level the structure of grains shows the random arrangement. We will consider the following assumptions:

Assumption I.: *Ensemble of samples which was made with the same technological procedure and has the same geometrical dimension and volume fractions or mass fractions of the individual components and shows the same values of their parameters on the macroscopic level.*
Assumption II.: *Composite materials on the macroscopic level are homogeneous and isotropic.*

In these cases the structure on the submacroscopic level can be determined only statistically [1]. The local values of parameters of composite materials are dependent on the space co-ordinates and there are also random quantities, and therefore the heat equation is a stochastic equation. Its solution is a formidable task. In such cases one is obliged to use the approximation methods.

The justification of the use of the phenomenological equation for description of the irreversible process requires that the geometrical dimensions of grains to be larger than the free path of the carries of energy and charge. If these conditions are fulfilled then the heat equation for the composite material has the following form

$$\rho(\mathbf{r})c(\mathbf{r})\frac{\partial T}{\partial t} = \nabla.\lambda(\mathbf{r})\nabla T \, , \tag{1}$$

where $\rho(\mathbf{r})$ is the density, $c(\mathbf{r})$ is the specific heat at constant pressure, $\lambda(\mathbf{r})$ is the thermal conductivity.

The experimenter uses standard methods for the determination of the parameters of the composite material on the macroscopic level. From this reason, it is very important for him to know in which cases the composite material on the macroscopic level may be characterised by effective parameters. Because only in these cases it is justifiable to use the standard methods for their measurement. The necessary and sufficient conditions for using effective parameters are discussed in [1]. In further text we will assume that the conditions for using effective parameters are fulfilled. In such cases, there is also a problem to determine how the effective parameters depend on the structure of the composite material on a submacroscopic level and also how they depend on the quantities, which characterise individual components of the particulate composite. Now we can formulate the aims of this paper. They are the following: To derive the effective heat equation and to derive the formula for the effective parameters.

The standard methods for determination of the thermophysical parameters of the particulate composite are based on the use of the following relation

$$< \mathbf{q}(\mathbf{r}) >= -\lambda_{eff} \nabla < T > \tag{2}$$

where $\mathbf{q}(\mathbf{r})$is the heat current density, $\lambda_{eff}$is the effective thermal conductivity, <> means the averaging of certain random quantity through a representative volume ΔV which is large enough in order to contain many grains and it must be very small compared to all dimensions of the specimen and perhaps to other lengths which are important in experiment . Our task will be to derive relation (2) and from it we obtain the formula for λ_{eff}.

2 Derivation of effective of heat equation for particulate composite

After an application of the Laplace's transformation to equation (1) we obtain

$$\gamma(\mathbf{r})\left[p\widetilde{T} - T(0,\mathbf{r})\right] = \nabla.\lambda(\mathbf{r})\nabla\widetilde{T} \tag{3}$$

where $\gamma(\mathbf{r}) = \rho(\mathbf{r})c(\mathbf{r})$, $\widetilde{T} = \int_0^\infty e^{-pt}T(t,\mathbf{r})dt$. We introduce the following equation

$$\gamma_p\left[p\widetilde{T}_p - T(0,\mathbf{r})\right] = \lambda_p \Delta \widetilde{T}_p \tag{4}$$

We will consider that the solution of equation (3) and (4) is given not only at the same initial conditions but also at the same boundary ones. According to relation (2) we want to show at which conditions $\widetilde{T}_p =< \widetilde{T} >$ and $\lambda_p = \lambda_{eff}$. For the following calculation it will be suitable to introduce the denotations $\lambda' = \lambda(\mathbf{r}) - \lambda_p$, $\gamma' = \gamma(\mathbf{r}) - \gamma_p$, $\widetilde{T}' = \widetilde{T} - \widetilde{T}_p$. With the help of equation (3) and (4) it may be show that

$$\widetilde{T}' = -\left[\Delta + \nabla.\frac{\lambda'}{\lambda_p}\nabla - \frac{\gamma}{\lambda_p}p\right]^{-1}\left\{\left[\nabla.\frac{\lambda''}{\lambda_p}\nabla - \frac{\gamma'}{\lambda_p}p\right]\widetilde{T}_p + \frac{\gamma'}{\lambda_p}T(0,\mathbf{r})\right\} \tag{5}$$

or

$$\widetilde{T} = \left[\Delta - \frac{\gamma_p}{\lambda_p}p + \nabla.\frac{\lambda'}{\lambda_p}\nabla - \frac{\gamma'}{\lambda_p}p\right]^{-1}\left\{\left[\Delta - \frac{\gamma_p}{\lambda_p}p\right]\widetilde{T}_p - \frac{\gamma'}{\lambda_p}T(0,\mathbf{r})\right\} \tag{6}$$

For further calculation we will use the following operator identity

$$\left[\hat{L}_0+\hat{L}_1\right]^{-1}=\sum_{n=0}^{\infty}(-1)^n\left\{\hat{L}_0^{-1}\hat{L}_1\right\}^n L_0^{-1} \tag{7}$$

According to (7) one can write

$$\left[\Delta-\frac{\gamma_p}{\lambda_p}p+\nabla.\frac{\lambda'}{\lambda_p}\nabla-\frac{\gamma'}{\lambda_p}p\right]^{-1}=\sum_{n=0}^{\infty}\begin{array}{l}\left\{\left[\Delta-\frac{\gamma_p}{\lambda_p}p+\nabla.\frac{\lambda'}{\lambda_p}\nabla\right]^{-1}\frac{\gamma'}{\lambda_p}p\right\}^n\times\\ \times\left[\Delta-\frac{\gamma_p}{\lambda_p}p+\nabla.\frac{\lambda'}{\lambda_p}\nabla\right]^{-1}\end{array} \tag{8}$$

and

$$\left[\Delta-\frac{\gamma_p}{\lambda_p}p+\nabla.\frac{\lambda'}{\lambda_p}\nabla\right]^{-1}=\hat{L}-\hat{\mathbf{L}}\frac{\lambda'}{\lambda_p}.\sum_{n=0}^{\infty}(-1)^n\left\{\hat{\mathsf{L}}\frac{\lambda'}{\lambda_p}\right\}^n.\hat{\mathbf{L}} \tag{9}$$

where

$$\hat{L}=\left[\Delta-\frac{\gamma_p}{\lambda_p}p\right]^{-1},\ \hat{\mathbf{L}}=\nabla\left[\Delta-\frac{\gamma_p}{\lambda_p}p\right]^{-1}\text{ and }\hat{\mathsf{L}}=\nabla\nabla\left[\Delta-\frac{\gamma_p}{\lambda_p}p\right]^{-1}.$$

We can divide three operators $\hat{L}$, $\hat{\mathbf{L}}$ and $\hat{\mathsf{L}}$ in two parts; the singular and regular one.

Standard methods of measuring the thermophysical parameters are based on local heat equation, and therefore if we want to obtain the local effective heat equation we have to use in (8) and (9) only the singular part of $\hat{L}$, $\tilde{\mathbf{L}}$ and $\hat{\mathsf{L}}$. In appendix it is shown that for the isotropic grains of globular shape the operators $\hat{L}^{\text{sing}}$, $\hat{\mathbf{L}}^{\text{sing}}$ and $\hat{\mathsf{L}}^{\text{sing}}$ are expressed by the following relations: $\hat{L}^{\sin g}=0$, $\hat{\mathbf{L}}^{\text{sing}}=0$ and $\hat{\mathsf{L}}^{\text{sing}}=a\mathsf{I}$. I is the unit tensor. In the framework of this approximation one can write

$$\tilde{T}\cong\tilde{T}_p-\hat{\mathbf{L}}\frac{\lambda'}{\lambda_{\mathbf{p}}}.\sum_{n=0}^{\infty}(-1)^n\left\{\hat{\mathsf{L}}^{\text{sing}}\frac{\lambda'}{\lambda_p}\right\}^n\nabla\tilde{T}_p \tag{10}$$

In (10) we leave $\hat{\mathbf{L}}$ what also in this case leads to local effective heat equation. Using singular part of operators $\hat{L}^{\sin g}$,$\hat{\mathbf{L}}^{\text{sing}}$ and $\hat{\mathsf{L}}^{\text{sing}}$ and (10) one obtains

$$\tilde{T}=\tilde{T}_p-\hat{\mathbf{L}}\frac{\lambda'}{\lambda_p}\left[1+a\frac{\lambda'}{\lambda_p}\right]^{-1}\nabla\tilde{T}_p \tag{11}$$

After averaging (11) one obtains $<\tilde{T}>=\tilde{T}_p$ because we put

$$<\frac{\lambda'}{\lambda_p}\left[1+a\frac{\lambda'}{\lambda_p}\right]^{-1}>=0. \tag{12}$$

If (2) has to be valid then we will calculate the following relation

$$<\tilde{\mathbf{q}}(\mathbf{r})>=-<\lambda(\mathbf{r})\nabla\tilde{T}>=-\lambda_p<\tilde{T}>, \tag{13}$$

where we used (11). Comparing (2) and (13) we obtain $\lambda_p=\lambda_{eff}$.It still remains for us to determine $<\gamma\tilde{T}>$. Using (10) we can write

$$<\gamma\tilde{T}>=<\gamma>\tilde{T}_p-<\gamma\tilde{\mathbf{L}}\frac{\lambda'}{\lambda_{\mathbf{p}}}\left[1+a\frac{\lambda'}{\lambda_p}\right]^{-1}>\nabla\tilde{T}_p \tag{14}$$

If we substitute the $\hat{\mathbf{L}}^{\text{sing}}$ instead $\hat{\mathbf{L}}$ in (14) we obtain

$$<\gamma\tilde{T}>=<\gamma>\tilde{T}_p=<\gamma><\tilde{T}> \tag{15}$$

Introducing (13) and (15) into (3) one obtains

$$<\gamma>\left[p<\tilde{T}>-T(0,\mathbf{r})\right]=\lambda_{eff}\Delta<\tilde{T}> \tag{16}$$

or

$$<\gamma>\frac{\partial<T>}{\partial t}=\lambda_{eff}\Delta<T> \tag{17}$$

Equation (17) represents the local effective heat equation.

3 Determination of $<\gamma>$ and λ_{eff}

Comparing equations (3) and (16) and using $\lambda_p=\lambda_{eff}$ we obtain that $<\lambda>=\gamma_p$. There is some difference between λ_p and γ_p. In regard to $\gamma_p=<\gamma>$ we see that $\lambda_p\neq<\lambda>$ because λ_{eff} depends on the paths of the transport of the energy.

According to assumption II. the probability that in a certain place there is located the ith component is expressed by volume fraction $\frac{V_i}{V}$ of the ith components. From this fact it follows that

$$<\gamma>=\sum_{i=1}^{n}\rho_i c_i(\mathbf{r})\frac{V_i}{V} \tag{18}$$

and

$$<\frac{\lambda''}{\lambda_p}\left[1+a\frac{\lambda'}{\lambda_p}\right]^{-1}>=\sum_{i=1}^{n}\frac{V_i}{V}\frac{\lambda_i-\lambda_{eff}}{(1-a)\lambda_{eff}+a\lambda_i}=0. \tag{19}$$

Finally we will interpretate relation (18). The average density is expressed by the relation

$$<\rho> = \sum_{i=1}^{n} \rho_i c_i = \sum_{i=1}^{n} \frac{M_i}{V_i}\frac{V_i}{V} = \frac{M}{V} \tag{20}$$

where $M = \sum_{i=1}^{n} M_i$ and M_i is the mass of i$^{\text{th}}$ component. Further

$$<c>_w = \sum_{i=1}^{n} c_i w_i, \tag{21}$$

where $w_i = \frac{M_i}{M}$ is the mass fraction of i$^{\text{th}}$ component. Introducing mass fractions into (21) one obtains

$$<c>_w = \sum_{i=1}^{n} \frac{Q_i}{M_i}\frac{M_i}{M} = \frac{Q}{M} \tag{22}$$

where Q_i is the heat needful for the increase of temperature of i$^{\text{th}}$ component about 1K. Now we can write

$$<\gamma> = \sum_{i=1}^{n} \rho_i c_i \frac{V_i}{V} = \sum_{i=1}^{n} \frac{M_i}{V_i}\frac{Q_i}{M_i}\frac{V_i}{V} = <\rho><c>_{w.} \tag{23}$$

Relation (23) is due to Assumption II.

4 Conclusion

The effective non-stationary heat equation and the formula for the effective thermal conductivity were derived.

Appendix

Operator $\hat{L}$ is defined in the following way: $\hat{L} f(\mathbf{r}) = \int G(\mathbf{r}-\mathbf{r}') f(\mathbf{r}') d\mathbf{r}'$. Applying operator $\left[\Delta - \frac{\gamma_p}{\lambda_p} p\right]$ we obtain $f(\mathbf{r}) = \int \left\lfloor \Delta - \frac{\gamma_p}{\lambda_p} p \right\rfloor G(\mathbf{r}-\mathbf{r}') f(\mathbf{r}') d\mathbf{r}'$. If the last equation has to be fulfilled then the following equation has to hold

$$\left\lfloor \Delta - \frac{\gamma_p}{\lambda_p} p \right\rfloor G(\mathbf{r}-\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}') \tag{24}$$

The solution of equation is expressed in form

$$G(\mathbf{r}) = -\frac{1}{4\pi}\frac{1}{r}\exp\left[-\sqrt{\frac{\gamma_p}{\lambda_p} pr}\right]$$

With the help of the function $G(\mathbf{r})$ the operators $\hat{L}^{\sin g}$, $\hat{\mathbf{L}}^{\mathrm{sing}}$ and $\hat{\mathsf{L}}^{\mathrm{sing}}$ are defined in the following way: $\hat{L}^{\sin g} \overset{def}{=} \lim_{R\to 0} \int_{\Omega_R} G(\mathbf{r})d\mathbf{r}\ \hat{1}$; $\hat{\mathbf{L}}^{\sin g} \overset{\mathrm{def}}{=} \lim_{R\to 0} \int_{\Omega_R} \nabla G(\mathbf{r})d\mathbf{r}\hat{1}$;

$\widetilde{\mathsf{L}}^{\mathrm{sing}} \overset{def}{=} \lim_{R\to 0} \int_{\Omega_R} \nabla\nabla G(\mathbf{r})d\mathbf{r}\hat{1}$; where Ω_R is the region of spherical shape. From the above definitions of the singular operators one can easy prove that $\hat{L}^{\sin g} = 0$, $\hat{\mathbf{L}}^{\mathrm{sing}} = 0$, $\hat{\mathsf{L}}^{\mathrm{sing}} = a\hat{\mathbf{I}}$, where $a = \frac{1}{3}$.

Acknowledgement

Authors wish to thank the Slovak Science Grant Agency for the financial support under the contract 4256/99.

References

[1] Beran M J, 1974 Application of Statistical Theories for the Determination of Thermal, Electrical and Magnetic Properties of Heterogenous Materials. In *Composite Materials, Vol.2.* (New York, Academic Press) p.209

MEASURING THE TEMPERATURE-DEPENDENT THERMAL CONDUCTIVITY AND HEAT CAPACITY IN BUILDING MATERIALS

Jozefa Lukovičová, Jozef Zámečník

Department of Physics, Faculty of Civil Engineering, Slovak Technical University,
Radlinského 11, SK-813 68 Bratislava, Slovakia
Email: lukovico@svf.stuba.sk, zamecnik@svf.stuba.sk

Abstract

The method for measuring the temperature dependent thermal conductivity and heat capacity is based on the solution of the nonlinear inverse problem of a parameter identification. The solution of the corresponding direct problem is obtained using a time marching boundary element method. The determination of this thermal properties requires boundary and initial data and a set of temperature measurements at a single sensor location inside the heat conducting body. The application of this method is illustrated for one dimensional heat conductivity in the building material furnance slag - based concrete.

Keywords: thermal conductivity, heat capacity, inverse problem, building materials

1 Introduction

Many theoretical and experimental methods for measuring the thermophysical properties are developed in the literature, they include, among others, the steady-state method, the probe method, the periodic heating method, the least squares method and pulse heating method. This paper deals with the method for measuring temperature-dependent thermal conductivity and heat capacity based on the inverse problem of identification of parameters. The method can be considered as a reasonable alternative to the classical methods for measuring thermal properties, because for wide temperature range it is possible to determine thermal properties as functions of temperature.

The determination of the mater parameters from some global measurements belongs to the class of inverse problems, which are known to be ill-posed. The effectiveness of the inverse solution is substantially dependent on numerical realization of the direct problem's solution and on its precision.

Theoretical studies of the identification of parameters were investigated by Cannon [1]. However, these studies, in order to make the identification heat conduction problem well-posed, introduce strong hypotheses on the input data which are seldom satisfied in practical experiments. Huang [2] employed a weighted finite difference method for the solution of the direct problem as part of the inverse problem that should be modified, for mixed boundary condition. Such modification is not required if the boundary element method (BEM) is used. A review of BEM can be found in Brebbia [3]. Ingham [4] implemented the BEM in the direct nonlinear case of the inverse identification situation. We use modified Ingham's idea.

2 Formulation of the problem

We consider the one-dimensional, nonlinear heat conduction problem in slab geometry. The dimensionless mathematical formulation of this problem can be expressed as

$$C(T)\frac{\partial T(x,t)}{\partial t}=F_0\frac{\partial}{\partial x}(k(T)\frac{\partial T(x,t)}{\partial x}) \quad (x,t)\in(0,1)\times(0,1] \tag{1a}$$

$$-k(T)\frac{\partial T(x,t)}{\partial x}=q_0(t) \quad \text{at } x=0, \quad t\in(0,1] \tag{1b}$$

$$T(x,t)=T_1(t) \quad \text{at } x=1, \quad t\in(0,1] \tag{1c}$$

$$T(x,t)=T_0(x) \quad \text{for } t=0, \quad x\in[0,1] \tag{1d}$$

where T is the temperature, $q=k(T)\frac{\partial T(x,t)}{\partial x}$ is the heat flux, $k(T)$ is the thermal conductivity, $C(T)$ is the heat capacity per unit volume, $q_0(t)$, $T_1(t)$, $T_0(x)$ are known functions. The temperature, distance, time, heat capacity and thermal conductivity are dimensionless with respect to T_r (a reference temperature), L (length of the slab), t_f (final time of interest during which a specific practical heat conduction experiment is performed), C_r, and k_r (reference value), respectively. The Fourier number $F_0=(k_r t_f)/(C_r L^2)$.

A temperature sensor is installed at an arbitrary spatial position $x=d\in(0,1)$ and temperature measurements $T^{(m)}(t)$ are recorded in time, namely

$$T(x,t)=T^{(m)}(t) \quad \text{at} \quad x=d, \quad t\in(0,1] \tag{2}$$

For the inverse problem, the thermal properties $k(T)$ and $C(T)$ are regarded as being unknown, but everything else in equation (1) is known. The determination of $k(T)$ and $C(T)$ from boundary and initial data and temperature measurements $T^{(m)}$ is needed to make by utilizing an inverse analysis.

3 Direct problem solution

The first step of inverse analysis is to develop the corresponding direct solution for the problem (1). A boundary element method is employed for solution of the nonlinear system (1) . By using the Kirchhoff transformation

$$u(T)=\int_0^T k(T)dT \tag{3}$$

denoting $u(x,t)=u(T(x,t))$, $(x,t)\in(0,1)\times[0,1]$, equations (1) in the new variable u can be written as

$$\frac{\partial u(x,t)}{\partial t}=a(T(x,t))\frac{\partial^2 u(x,t)}{\partial x^2}, \quad (x,t)\in(0,1)\times(0,1] \tag{4a}$$

$$-\frac{\partial u(x,t)}{\partial x}=u(q_0(t)) \quad \text{at} \quad x=0, \quad t\in(0,1] \tag{4b}$$

$$u(x,t) = u(T_1(t)) \qquad \text{at } \ x = 1, \qquad t \in (0,1] \tag{4c}$$

$$u(x,t) = u(T_0(x)) \quad \text{for } \ t = 0, \qquad x \in [0,1] \tag{4d}$$

where

$$a(T) = F_0 \frac{k(T)}{C(T)} \tag{5}$$

is the dimensionless thermal diffusivity.

According to weighted residuals method, the residue of (4a) weighted with fundamental solution u^* and integrated over the domain produces zero

$$\iint_{tt_f L} (a(T)\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t})u^* dx d\tau = 0 \tag{6}$$

If the thermal diffusivity a is constant, then for (4a) a fundamental solution is available

$$u^*(x,t;\xi,\tau) = \frac{1}{(4\pi a(t-\tau))^{\frac{1}{2}}} \exp(-\frac{(x-\xi)^2}{4a(t-\tau)})H(t-\tau)$$

where H is the Heaviside function and ξ and τ are generic space and time variables, respectively. In the case of non-constant thermal diffusivity the use of the fundamental solution (6) is accompanied by a time marching technique in which $a(T)$ is assumed constant at the beginning of each time step. Therefore, starting from the initial time $t_0 = 0$, over each time element $[t_{i-1}, t_i]$, the value of a_i is taken as the mean average

$$a_i = \overline{a(T)} = \int_0^1 a(T(x,t_{i-1}))dx \tag{7}$$

Therefore, over each time step nonlinear partial differential equation is linearized. Applying the Gaussian reciprocity theorem, using the fundamental solution u^* and approximation (7), equation (6) is transformed into following integral equation for each time step $[t_{i-1}, t_i]$, see Brebbia.[5]

$$\begin{aligned} \eta(x)u(x,t) = {} & \int_{i-1}^{i} a_i u'(0,\tau)u^*(x,t;0,\tau)d\tau + \int_{t_{i-1}}^{t_i} a_i u'(1,\tau)u^*(x,t;1,\tau)d\tau \ - \\ & \int_{t_i}^{t_i} a_i u(0,\tau)u^{*'}(x,t;0,\tau)d\tau - \int_{t_{i-1}}^{t_i} a_i u(1,\tau)u^{*'}(x,t;1,\tau)d\tau + \\ & \sum_{j=1}^{N_0} u(y,t_{i-1}) \int_{y_{j-1}}^{y_j} u^*(x,t,y,t_{i-1})dy, \quad t \in [t_{i-1},t_i], \quad x,y \in (0,1) \end{aligned} \tag{8}$$

where primes denote differentiation with respect to x and $\eta(x)$ is a coefficient which is equal to1 for $x \in (0,1)$ and 0.5 if $x \in \{0,1\}$.

Assuming that the temperature and the heat flux are constant over each time step $[t_{i-1}, t_i]$, the constant approximation of integral equation (8) and of equations (4b) and (4c), can be written in the form

$$\eta(x)u(x,\tilde{t}_i) = u'(0,\tilde{t}_i)\int_{i-1}^{i} a_i u^*(x,\tilde{t}_i;0,\tau)d\tau + u'(1,\tilde{t}_i)\int_{t_{i-1}}^{t_i} a_i u^*(x,\tilde{t}_i;1,\tau)d\tau \ -$$

$$u(0,\tilde{t}_i)\int_{t_i}^{t_i} a_i u^{*'}(x,\tilde{t}_i;0,\tau)d\tau - u(1,\tilde{t}_i)\int_{t_{i-1}}^{t_i} a_i u^{*'}(x,\tilde{t}_i;1,\tau)d\tau +$$

$$\sum_{j=1}^{N_0} u(\tilde{y}_j,\tilde{t}_{i-1})\int_{y_{j-1}}^{y_j} u(x,\tilde{t}_i,y,t_{i-1})dy \quad , \quad x,y \in (0,1) \tag{9a}$$

$$-u'(0,\tilde{t}_i) = u(q_0(t_i)), \qquad t \in [t_{i-1},t_i] \tag{9b}$$

$$u(1,\tilde{t}_i) = u(T_0(1,\tilde{t}_i)), \qquad t \in [t_{i-1},t_i] \tag{9c}$$

where $\tilde{t}_i = (t_{i-1} + t_i)/2$ is midpoint of the element $|t_{i-1},t_i|$ and $\tilde{y}_j = (y_{j-1} + y_j)/2$ for $j = 1$ to N, are the midpoints of the elements $|y_{j-1}, y_j|$ which are used to discretise the segment $[0,1]$ into N elements. The integrals in (9) are calculated analytically.

For calculating temperature function $u(x,t)$ at any point inside the layer $[0,1]\times[t_{i-1},t_i]$ it is needed to find heat flux $u'(0,\tilde{t}_i)$, $u'(1,\tilde{t}_i)$ and transformed temperature $u(0,\tilde{t}_i)$, $u(1,\tilde{t}_i)$ on the boundaries $x = 0$ and $x = 1$, which can be obtained by solving a system of four equations (9b,c) and (9a)- if point x tend to 0 and to 1. Once the values of $u(x,t)$ are obtained, the temperature $T(x,t)$ is calculated by inverting the transformation (3), namely

$$T(x,t) = u^{-1}(u(x,t)), \qquad (x,t) \in [0,1]\times[t_{i-1}.t_i] \tag{10}$$

and boundary heat fluxes are given by

$$q(x,t) = q(x,\tilde{t}_i) = u'(x,\tilde{t}_i) \quad x \in \{0,1\}, \; t \in [t_{i-1},t_i] \tag{11}$$

In particular, the values of $u(\tilde{y}_j,t_i)$ for $j = 1$ to N, need to be calculated in order to provide the ‘initial’ condition at the time t_i and to proceed to the next time step $[t_i,t_{i+1}]$. Also the corresponding values of the temperature, $T(\tilde{y}_j,t_i)$ for $j = 1$ to N, are required in order to calculate the new constant value of the thermal diffusiviy a_{i+1}, given by (7) at the time t_i .Based on this time marching technique the BEM provides the values of u and C at any point in the solution domain.

4 Inverse problem

For the inverse problem, the thermal properties $k(T)$ and $C(T)$ are regarded as being unknown, but everything else in equations (1) is known. In addition temperature readings $T^{(m)}(t)$ taken at arbitrary spatial position $x = d \in (0,1)$ are considered available.

For a given $k(T)$ and $C(T)$ (initial guesses of $k(T)$ and $C(T)$) is denoted the solution of (9) by $T(x,t;C,k)$ and the solution at $x = d$, $T^{(c)}(t)$. The solution of the present inverse problem is to be obtained in such a way that the least-squares norm $\left\|T^{(c)} - T^{(m)}\right\|^2$ is minimized. In practice, only a finite set of time measurements may be available at some discrete times, t_i', namely

$$T(d,t_i') = T^{(m)}(t_i') = T_i^{(m)}, \; i = 1 \text{ to } M \tag{12}$$

Requiring that the continuous functions *k*(*T*) and *C*(*T*) be determined from only a finite set of data (12) results in a non-unique solution problem. In order to be able to achieve a unique solution, the unknown functions *k*(*T*) and *C*(*T*) are parameterized. The parameterization is performed by assuming that the *k*(*T*) and *C*(*T*) are taken as a set of polynomials,

$$C(T)=\sum_{j=1}^{R} C_j T^{j-1} \text{ and } \quad k(T)=\sum_{j=1}^{R} k_j T^{j-1} \tag{13}$$

Then the least-squares norm in discretised form becomes

$$S(\underline{C},\underline{k})=\sum_{i=1}^{M}[T_i^{(m)}-T_i^{(c)}(\underline{C},\underline{k})]^2 \tag{14}$$

where $\underline{C}=(C_j)$ and $\underline{k}=(k_j)$, for $j=1$ to R, are the unknown vectors of the thermal conductivity and heat capacity, and $T_i^{(c)}(\underline{C},\underline{k})$ is the calculated value of the temperature at $t=t_i^{\prime}$, for $i=1$ to M, obtained from the BEM solution of the direct problem (9), by using the estimated values of the $(\underline{C},\underline{k})$. The unknown parameters k_j and C_j are then determined as the solution of the minimization nonlinear least-squares norm (13) using the Newton -Raphson method, by the package LSODA[6].

Temperature dependent uniqueness conditions applicable to the problem of estimating $C(T)$ and $k(T)$ are very difficult, see Cannon [1] and as simple alternative, in order to be able to obtain a unique solution, $C(T)$ and $k(T)$ are fixed at some points. The stability oft the solution is ensured since the number of independent parameters which are to be estimated is, in general, small and no further regularization terms is needed.

6 Experimental results

The experiments were performed on the furnace-slag-based concrete. The measured sample is an alloy geometry of length $L=0.2$ m, with reference thermal conductivity $k_r=10^{-1}\,\mathrm{W/mK}$, heat capacity $C_r=10^6\,\mathrm{J/m^3K}$, temperature $T_r=T_{\max}-T_0=180$ °C and is subject to a heat transfer experiment with $q_0(t)=0.2T$ and $T_1(t)=T_0=20/180$ over a period of time $t_f=3.6\times10^3$s, the Fourier number $F_0=0.9$. The heat flux $q_0(t)$ is known at a prescribed rate of $\Delta t^*=t_f\Delta t$, temperature $T^{(m)}$ is recorded at sampling rate of $\Delta t'^*=t_f/M$.

The inverse determination of the heat capacity and the thermal conductivity starting initially with the guesses:

$$C(T)=C_1+C_2T+C_3T^2 \qquad \text{and} \qquad k(T)=k_1+k_2T+k_3T^2$$

where $C_1=1$, $C_2=0.75$, $C_3=1.25$, $k_2=4$, $k_3=4.75$
For the time step $\Delta t=0.01$ and number of measurements $M=60$, numerical dimensionless results are obtained as follows:

$$C(T)=1.3+0.203\,T+0.6015\,T^2 \qquad \text{and} \qquad k(T)=3.4+1.1821\,T+3.4521\,T^2$$

Real physical values are shown on Fig.1

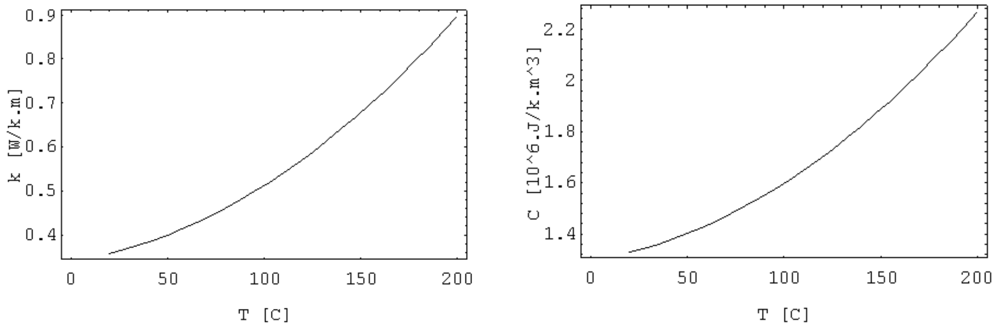


Fig 1 Thermal conductivity and heat capacity as a function of temperature

Precise tests of accuracy and stability of the method as well as the sensitivity to the experimental errors are tested by Lukovičová and Zámečník in [7].

7 Conclusion

The presented method for simultaneous determining the thermal conductivity and heat capacity as a function of temperature is found by means of the solution of an inverse heat conduction problem. The method is suited for practical calculating of thermal properties of porous materials and for analyzing of heat conduction measurement problems.

Acknowledgment

Authors are grateful to VEGA (Grant No.1/4063/79) for financial support of this work.

References

[1] Cannon J R, 1964 Determination of certain parameters in heat conduction problems, *J. Math. Anal. Appl.* **8**, 188-201

[2] Huang C H, Yan J Y, 1995 An inverse problem in simultaneously measuring temperature-dependent thermal conductivity and heat capacity, *Int. J. Heat Mass Transfer* **38**, 3433-3441

[3] Brebbia C A, 1984 Application of the boundary element method for heat transfer problems, in *Proc.Conf. Model et Simulation en Thermique* (Poitiers:Ensma) p.1

[4] Ingham D B, Yuan Y, 1993 The solution of nonlinear inverse problem in heat transfer, *IMA J. Appl. Math.* **50**, 113-132

[5] Brebbia C A, Telles J C F, Wrobel L C, 1984 *Boundary element Techniques: Theory and Application in Engineering,* (Berlin: Springer-Verlag)

[6] Constales D, Automatic differentiation in lsoda. *In preparation*

[7] Lukovičová J, Zámečník J, 1999 Determinination of thermophysical coefficients in nonlinear heat conducting porous material, *Slovak J. Civil Eng.*, In preparation

THE LINEAR THERMAL EXPANSION OF BUILDING MATERIALS AT HIGH TEMPERATURES

Jan Toman[1], **Robert Černý**[2]

[1] Department of Physics, Faculty of Civil Engineering, Czech Technical University, Thákurova 7, 166 29 Prague 6, Czech Republic
[2] Department of Structural Mechanics, Faculty of Civil Engineering, Czech Technical University, Thákurova 7, 166 29 Prague 6, Czech Republic
Email: toman@fsv.cvut.cz, cernyr@fsv.cvut.cz

Abstract

A method for measuring linear thermal expansion of porous materials in the high temperature range up to 1000°C is introduced in the paper. The measuring device is based on the application of a comparative technique. A bar sample of the studied material is put into a cylindrical, vertically oriented electric furnace. As it is technically difficult to perform length measurements directly in the furnace, a thin ceramic rod, which passes through the furnace cover is fixed on the top side of the measured sample. The length changes can be determined outside the furnace in this way provided the measurement is performed at the same time on the sample of a standard material. A practical application of the method is demonstrated with two types of common porous building materials, cement mortar and refractory concrete.

Key words: thermal expansion, high temperatures, building materials

1 Introduction

Thermal expansion of solid materials is measured by commercially produced dilatometers mostly. Various treatments are employed, for instance the methods based on variations of electric resistance, capacity, inductance or the interference methods (see for instance [1-3] for details). Among most commonly used laboratory devices for measuring thermal dilatation of bar solid samples belong Edelmann dilatometer and various comparators.

The most of the mentioned methods are suitable for relatively small samples, typically up to 10 mm in standard experimental setups. For this reason, they can be applied for measuring metals, glass, various polymers and other materials which are homogeneous in this length scale, but for porous building materials with characteristic dimensions of nonhomogeneities as high as several mm or cm are they practically inapplicable. Therefore, some modifications of standard treatments are necessary in order to achieve a sufficient accuracy in measuring thermal expansion of these materials.

Two examples of such modifications are described in [4]. The first of them is based on the application of a contact comparator, the second one employs an optical comparative treatment and consists in measuring the positions of microtargets on the samples through the window of the climatizing chamber using cathetometric reading.

Both methods are supposed to measure linear thermal expansion coeffiient in the temperature range of -50°C to 200°C.

In this paper we introduce a method for measuring linear thermal expansion of porous building materials in the high temperature range, up to 1000°C in the current experimental setup.

2 Theoretical relations

The infinitesimal change of length due to the change of temperature is defined by

$$dl = l_0 \alpha dT \quad , \tag{1}$$

where l_o is the length at the reference temperature T_o, α is the linear thermal expansion coefficient.

From (1) it follows that

$$\alpha = \frac{1}{l_0}\frac{dl}{dT} \quad ; \tag{2}$$

and is generally a function of temperature, $\alpha = \alpha(T)$. Assuming $l_o = const.$ we obtain another useful relation from Eq. (2),

$$\alpha = \frac{d\left(\frac{l}{l_0}\right)}{dT} = \frac{d\varepsilon}{dT} \quad , \tag{3}$$

where ε is the relative elongation, $\varepsilon = \varepsilon\,(T)$.

Integrating (3) we arrive at

$$\varepsilon(T) = \int_{T_0}^{T} \alpha(\tau) d\tau \quad . \tag{4}$$

In small temperature intervals, where $T \to T_o$ we can assume

$$\alpha = \alpha_0 = const. \quad , \tag{5}$$

$$\alpha_0 = \frac{\varepsilon}{T - T_0} \quad , \tag{6}$$

and α_o can be calculated from a single experiment consisting in heating the sample from the initial temperature T_o to the final temperature T and measuring the relative elongation ε.

In wider temperature ranges, the assumption (5) is no longer valid, and we have to employ the general definition relation (3). We choose a reference temperature T_o, heat the samples to the temperatures T_i, $i = 1,...,n$ covering the desired temperature range and determine the corresponding values ε_i; $i = 1,...,n$. The pointwise given function $\varepsilon_i = f(T_i)$ is then approximated using a regression analysis and its continuous representation $\varepsilon\,(T)$ is obtained. Finally, the linear thermal expansion coefficient $\alpha(T)$ is calculated using (3) as the first derivative of the $\varepsilon(T)$ function with respect to temperature.

3 Experimental setup

The measuring device for determining the linear thermal expansion of porous materials in high temperature range is based on the application of a comparative technique. A bar sample of the studied material is put into a cylindrical, vertically oriented electric furnace. As it is technically difficult to perform length measurements directly in the furnace, a thin ceramic rod, which passes through the furnace cover, is fixed on the top side of the measured sample. The length changes can be determined outside the furnace in this way, for instance by a dial indicator, but on the other hand, the temperature field in the ceramic rod is very badly defined, and it is not possible to determine directly, which part of the total change of length is due to the measured sample and due to the ceramic rod.

Therefore, the measurement is performed at the same time on the sample of a standard material (such as copper or iron where the $\alpha(T)$ function is known) which is put into the furnace together with the studied material and is provided with an identical ceramic rod passing through the cover. The change of length of the ceramic rod can be determined in this way, and consequently also the length change of the measured sample.

It should be noted that temperature field is not constant in the whole volume of the furnace due to the differences of heat loss in the heated walls and in the cover. Therefore, temperature field in the furnace is measured by thermocouples, and an average value of temperature is considered in the α calculations.

4 Practical application

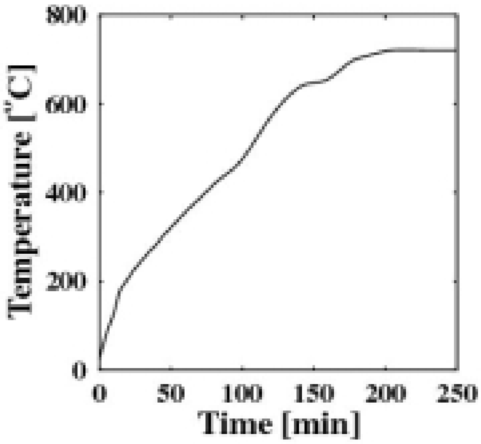
A practical measurement of the linear thermal expansion coefficient of a porous material on the device proposed in the previous Section can be described as follows. The measured sample and the standard are put into the furnace, provided by the contact ceramic rods, and the initial reading on the dial indicators is performed. Then, the power of the electric heating is adjusted for the desired temperature T_i in the furnace using a regulating transformer, and the length changes are monitored on the dial indicators. After the steady state is achieved, i.e. no temperature changes in the furnace and no length changes of both measured sample and the standard are observed, the final readings of length changes are done. The length change of the measured sample is calculated from the formula

$$\Delta l(T_i) = \Delta l_m(T_i) - \Delta l_s(T_i) + l_{\mathrm{o,s}} \int_{T_0}^{T_i} \alpha_s(\tau)\, d\tau \tag{7}$$

where Δl_m, Δl_s are the final readings of total length changes of the studied material and of the standard including the length changes of the ceramic rods, respectively, $l_{\mathrm{o,s}}$ is the initial length of the standard, α_s is the known linear thermal expansion coefficient of the standard, and the corresponding value of relative elongation can be expressed in the form

$$\varepsilon(T_i) = \frac{\Delta l(T_i)}{l_{\mathrm{o,m}}} \quad , \tag{8}$$

where $l_{\mathrm{o,m}}$ is the initial length of the measured sample.



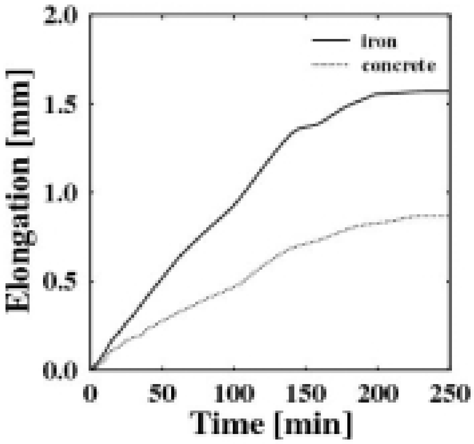


Fig 1 The dependence of the temperature in the furnace on time

Fig 2 The dependence of the elongation of iron and refractory concrete on time

The measurements are then repeated with other chosen values of furnace temperatures T_i, and the calculation of the $\alpha(T)$ function of the measured material is performed as explained in Section 2.

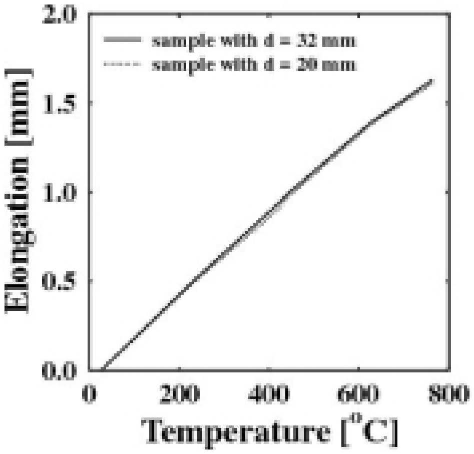
5 Experimental results

At first, a series of experimental measurements was performed to verify the designed method. Not only parameters of temperature and length changes, but also time dependencies of these quantities were studied. Examples of these dependencies are shown in Figs 1,2. The time necessary for the stabilization of the measuring device can be identified from these figures. In the case of $T = 700^oC$ it was approximately 4 hours.

Then, set of comparative experiments on one hand with two samples of the same material, and on the other hand with two different standard materials was done. Fig 3 shows a comparison of length changes of two cylindrical iron bars with two different diameters in dependence on temperature. Apparently, the length changes of both samples are very close each other, the maximum difference being 2%.

In the second part of our measurements we have performed experiments with the samples of two different types of porous building materials, cement mortar and refractory concrete. While cement mortar is designed for using in normal temperature conditions, the refractory concrete is supposed to be used in blast furnaces, and therefore it should have not only high temperature resistance, but also low thermal expansion coefficient in wide temperature range.

Fig 4 shows that the changes of linear thermal expansion coefficient of cement mortar with temperature are in a reasonable range only up to ~ 200^{o}C, then a dramatic course of the $\alpha(T)$ function with a maximum at ~ 500^{o}C followed by a fast decrease can be observed. Apparently, the samples of cement mortar do not resist to high temperatures very much. Structural changes and chemical reactions take place in the material if the temperatures grow higher than to ~ 200^{o}C, and the original structure is damaged. On the other hand, the refractory concrete was proved to be a suitable material for its designed application. Fig 5 shows that its linear thermal expansion



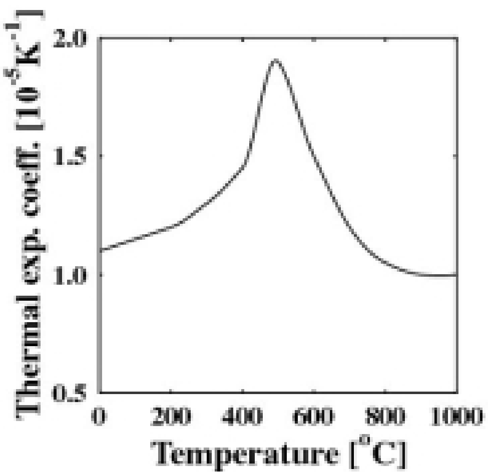


Fig 3 The dependence of the elongation of two iron samples with the different diameters *d* on temperature

Fig 4 The dependence of the linear thermal expansion coefficient of cement mortar on temperature

coefficient is significantly lower than for cement mortar already in the lower temperature range, and the increase of with temperature is relatively very slow.

6 Conclusions

A simple comparative method for measuring high temperature linear thermal expansion coefficient of porous materials which is suitable for application with relatively large samples up to 120 mm long was designed. A basic verification of the proper function and reliability of the method was done by the measurements with standard materials with the known $\alpha(T)$ functions. The practical measurements on cement mortar and refractory concrete have shown a good practical applicability of the proposed method which is significantly cheaper compared to the available commercial devices. The method can provide useful information for example in the design of building structures from the point of view of the fire security.

Acknowledgement

This paper is based upon work supported by the Grant Agency of the Czech Republic, under grants # 103/97/0094 and 103/97/K003.

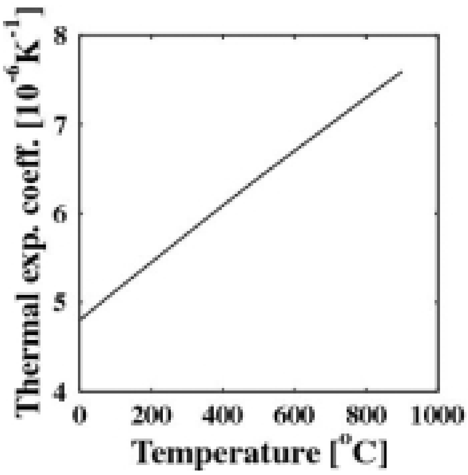


Fig 5 The dependence of the linear thermal expansion coefficient of refractory concrete on temperature

References

[1] Brož J et al, 1983 *Fundamentals of Physical Measurements* (in Czech, Prague: SPN)

[2] Gobrecht H et al, 1990 *Bergman-Schaefer Lehrbuch der Experimentalphysik, Band I, 10th Edition* (New York: Walter de Gruyter)

[3] Horák J, 1958 *Practical Physics* (in Czech, Prague: SNTL)

[4] Toman J, Černý R, 1996 Coupled Thermal and Moisture Expansion of Porous Materials, *Int. J. Thermophysics* **17**, 271-277

MEASUREMENT OF THERMOPHYSICAL PROPERTIES OF MATERIALS BY THE DYNAMIC PLANE SOURCE METHOD

Gabriela Labudová

Department of Physics, Faculty of Natural Sciences, Constantine the Philosopher
University, Tr. A. Hlinku 1, SK-94974 Nitra, Slovakia
Email: glabudova@ukf.sk

Abstract

The paper deals with the measurement of effusivity of materials using the dynamic plane source (DPS) method. The measurement of effusivity has been performed on air at room temperature on samples made from organic glass.

Key words: dynamic plane source method, adiabatic and isothermal conditions, effusivity

1 Introduction

Nowadays a lot of measuring methods for measurement thermophysical properties of materials have appeared in literature. Among others the dynamic plane source method (DPS) based on using an ideal plane sensor (PS). The PS sensor acts both as heat source and temperature detector. The dynamic plane source method is arranged for a one–dimensional heat flow into a finite sample. The outer (rear) surface of the sample is in contact with a poor heat conducting material so that the boundary condition of the sample is close to being adiabatic. In particular, the adiabatic method appears to be useful especially for measurement of materials with thermal conductivity in the range $2<\lambda<200$ W.m^{-1}.K^{-1}. The experimental arrangement is modified by exchanging the insulating material on the rear surface of the sample with a very good heat conducting material (heat sink) which makes the measurement possible also for samples with $\lambda\leq 2$ W.m^{-1}.K^{-1}. The presence of the heat sink on the rear surface of the sample makes that the heat conduction process through the sample after a defined time period approaches the steady – state condition. The adiabatic and the isothermal method give effusivity of materials, which includes information about thermal conductivity and thermal diffusivity of investigated material.

2 Theory

The theory considers ideal experimental conditions - the ideal heater (negligible thickness and mass), perfect thermal contact between PS sensor and the sample, zero thermal resistance between the sample and the material surrounding the sample, zero heat losses from the lateral surfaces of the sample [1].

If q is the total output of power per unit area dissipated by the heater, then the temperature increase as a function of time is given by [2]

$$T(x,t) = 2\frac{q\sqrt{a\,t}}{\lambda}\,\mathrm{ierf}\left(\frac{x}{2\sqrt{a\,t}}\right) \quad , \tag{1}$$

where a is the thermal diffusivity, λ the thermal conductivity of the sample and ierfc the error function [3].

We consider the PS sensor, which is placed between two identical samples having the same cross section as the sensor in the plane $x = 0$. The temperature increase in the sample as a function of time conforms

$$T(0,t) = \frac{q\sqrt{a}}{\lambda\sqrt{\pi}}\sqrt{t} \quad , \tag{2}$$

which corresponds to the linear heat flow into an infinite medium [3,4]. The slope of the graph of ΔT(t) against $\sqrt{t}$ gives the effusivity of the sample [5]

$$e = \frac{\lambda}{\sqrt{a}} = \sqrt{\rho c \lambda} \tag{3}$$

The real experiment can contribute to the deviation of the experimental curve from the ideal one [6]. Some of the distortions can be eliminated by the proper choice of the evaluation time interval while the others require further modifications of Eq. (1).

3 Experimental arrangement

Between two identical samples which forms a disk with the diameter of 3.10^{-2} m the PS sensor having the same cross section as the samples is placed. The sensor is made of a Ni – foil, 23 μm thick protected from both sides by an insulating layer made of kapton of 25 μm thick (Institute of Physics, SAS Bratislava). The sensor acts both as heat source and temperature detector, so the experimental arrangement is considerably simplified (Fig. 1) [7]. Providing adiabatic DPS method we have used polyurethane foam as a relatively good heat insulator for the sample, as the adjacent material and providing isothermal DPS method the samples were connected to a thermal sink, were made of Al blocks. The effusivity of the samples is obtained from the time history of the temperature rise of the sensor during an application of the electrical current. The temperature changes of the sensor result in changes of its resistance and, hence the voltage ΔU(t) across it accords to

$$\Delta U(t) = I_0\, R_0\, \alpha\, \Delta T(t) \tag{4}$$

Here I_0 is the current flowing through the PS sensor since $t = 0$ s, R_0 is the initial resistance of the PS sensor and α is the temperature coefficient of resistivity of the nickel the PS sensor consists from.

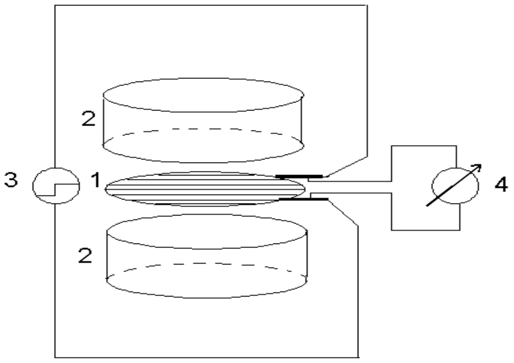


Fig. 1. The experimental setup (1 – PS sensor, 2 – samples, 3 – current source, 4 – milivoltmeter)

4 Results and discussion

The adiabatic and the isothermal DPS method has been tested on the samples of organic glass. The measurement of the effusivity has been performed on air at room temperature. The resistance of the Ni – heating element was about 1,221 Ω and the temperature coefficient of resistivity of the heater was 0,0047 K^{-1}. The effusivity of the sample was obtained from the slope of the graph $\Delta T(t)$ against $\sqrt{t}$ according to the Eq. (2). The experimental garphics were compared with the theoretical graphics and we investigated that for short time are this graphics identical, in Fig. 2. a), b) .

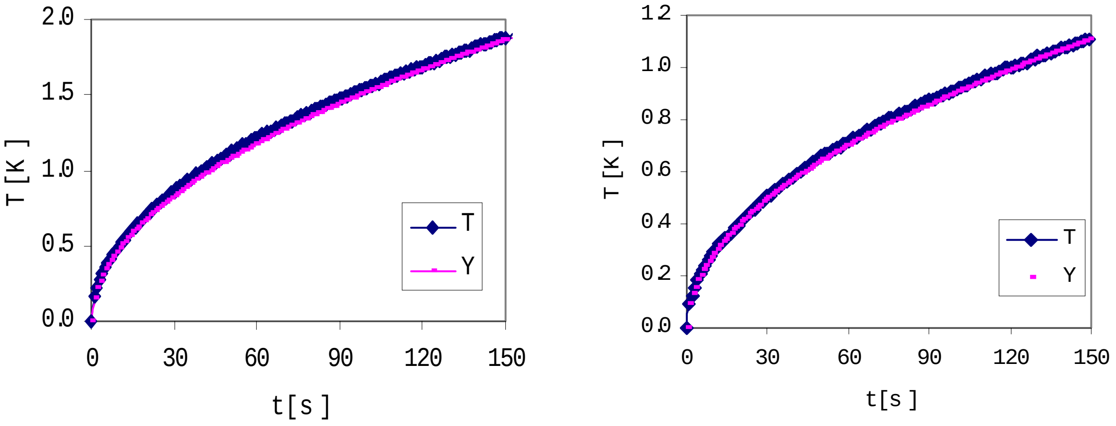


Fig. 2 Experimental (T) and theoretical (Y) dependence of the temperature increase $\Delta T(t)$ as a function of the time, a) adiabatic method, b) isothermal method.

The results of the measurements are presented in Table 1. The values are the mean values of ten independent measurements on several organic glass samples.

From the comparison of the experimental results and the effusivity, which is calculated from the thermal conductivity and the thermal diffusivity of the material from tabular data we can see that both adiabatic and isothermal DPS method give very close results.

Table 1. Experimental results (L - the sample length, e_1 (e_2) - the effusivity obtained from adiabatic (isothermal) DPS method, e_3 - the effusivity calculated from tabular data of thermal conductivity and thermal diffusivity , I - the current, diff e_1 (e_3), diff e_2(e_3) – the difference between the measured and calculated effusivities).

Mate-rial	L m	e_1 $\mathrm{Ws^{1/2}m^{-2}K^{-1}}$	e_2 $\mathrm{Ws^{1/2}m^{-2}K^{-1}}$	I A	e_3 $\mathrm{Ws^{1/2}m^{-2}K^{-1}}$	diff e_1(e_3) %	diff e_2(e_3) %
Org. glass	10^{-2}	568.8 +/-3.0	571.1 +/-3.7	0.267	570.73	0.333	0.061

Acknowledgement

Author wishes to thank the Slovak Science Grant Agency and the ministry of Education for the financial support under the contracts 1/6115/99 and Gr/Sl/L80.

References

[1] Karawacki E, Suleiman, B M, ul-Hag I, Nhi B T, 1992 An extension to the Dynamic Plane Source Technique for Measuring Thermal Conductivity, Thermal Diffusivity and Specific Heat of Dielectric Solids, *Rev. Sci. Instrum.*, **63**, 4390-4397

[2] Beck J V, Arnold K J, *Parameter Estimation in Engineering and Science*, (New York, John Wiley and Sons) p.448

[3] Carslaw H S., Jaeger J C, 1959 *Conduction of Heat in Solids*, (Oxford, Clarendon Press)

[4] Karawacki E, Suleiman B M, 1991 Dynamic Plane-Source Technique for the Study of the Thermal Transport Properties of Solids, *High Temp. High Press*. **23,** 215-223

[5] Kubičár L, Boháč V, 1997 Review of Several Dynamic Methods of Measuring Thermophysical Parameters, in *Thermal Conductivity 24*, (Pittsburgh, Technomic Publication Company, Inc.) p.135

[6] Kubičár L, 1990 *Pulse Method of Measuring Basic Thermophysical Parameters, Comprehensive Analytical Chemistry, Vol XII, Thermal Analysis, Part E,* (Ed: Svehla G, Amsterdam, Oxford, New York, Tokyo, Elsevier)

[7] Labudová G, Malinarič S, Vozár L, 1999 A Simple Apparatus for Measurement of Thermophysical Properties of Composites, in *Proceedings of the 9th Joint Seminar Development of Materials Science in Research and Education*, (Bratislava, STU) p.46

AN ACCURACY OF MEASUREMENT THE THERMAL DIFFUSIVITY OF TWO-LAYERED COMPOSITE

Libor Vozár[1], Wolfgang Hohenauer[2]

[1] Department of Physics, Faculty of Natural Sciences, Constantine the Philosopher University, Tr. A. Hlinku 1, SK-94974 Nitra, Slovakia
[2] Department of Materials Technology, Austrian Research Centers, A-2444 Seibersdorf, Austria
Email: vozar@uvt.uniag.sk, wolfgang.hohenauer@arcs.ac.at

Abstract

The paper deals with an investigation of two-layered systems using the laser flash method. The attention is dedicated to the study of the composites with the ideal thermal contact stated by zero thermal contact resistance.

It is well known that an estimation of the thermal diffusivity of a layer in a composite requires besides the knowledge of other relevant properties to know the thermal diffusivity of the remained layer. The inaccuracy of this 'known' thermal diffusivity significantly influences the accuracy of the unknown thermal diffusivity estimation. The paper summarizes the performed simulations that indicate error propagation and it guess the accuracy and measurement limits on an ideal two-layered system.

Key words: layered structures, thermal diffusivity, thermal contact resistance, flash method

1 Introduction

An application of various composites, especially ‚two-layered' coatings on substrates, has increased in a number of applications, such as thermal barriers, emissivity controls, electric insulation and wear, and erosion and corrosion resistance protection. Because such systems are being utilized under different thermal conditions, the knowledge of their thermophysical properties is of a great importance. In situ measurement on layered systems allow to test how thermophysical properties (the thermal conductivity and the thermal diffusivity) of components differ from those values received for bulk materials.

Determination of the thermal diffusivity of a component on a two-layered composite using the flash method can be viewed as a dependent measurement [1]. An estimation of the thermal diffusivity of one layer besides the knowledge of other relevant properties (the density, the heat capacity and the thickness of components) requires to know the thermal diffusivity of the remained layer. Errors in measurement of these 'additional' parameters are propagated through the data reduction and result as inaccuracy of the thermal diffusivity determination. This phenomenon influences the measurement limits that exist for an experimental apparatus and sample dimensions also in the case of measurement of homogeneous materials.

This paper presents results of the study of the influence of the inaccuracy of the 'known' thermal diffusivity to the accuracy of the unknown thermal diffusivity estimation. The methodology proposed in [1] is here applied to copper/alumina based two-layered systems.

2 The flash method

In the flash method, the front face of a disk-shaped sample is subjected to a pulse of radiant energy coming from a laser [2]. If material boundaries are flat and parallel to the sample front and rear surfaces and if there are no heat losses from the radial surface one-dimensional heat transfer occurs across the sample. Analyzing the resulting temperature rise on the opposite (rear) face of the sample any thermophysical property value (thermal diffusivity of one layer, or the thermal contact resistance) can be computed [3,4].

2.1 Theory

The analytical model assumes one-dimensional heat flow through the two-layered sample that consists of two layers of thickness e_1 and e_2. We consider a good thermal contact between the layers stated by zero thermal contact resistance. We assume to have uniform and constant thermal properties and densities of both layers and that the front face (at $x=-e_1$) is uniformly subjected to the instantaneous heat pulse with the heat Q supplied to the unit area. In case of non-ideal experimental conditions - when heat losses from the front and rear face should be taken into account, appropriate boundary conditions include heat flux terms with the heat transfer coefficients h_1 and h_2. The expression for the transient temperature rise at the sample rear face ($x=-e_2$) can be written in the non-dimensional normalized form [5,1] as

$$\theta(t) = \frac{T(t)}{T_\infty} = 2(1+PU)\sum_{\mathrm{k}=1}^{\infty} \frac{\exp\left(-\frac{\gamma_\mathrm{k}^2 t}{\omega_1^2 \eta_2^2}\right)}{d_0 + d_e} \quad . \tag{1}$$

Here T_∞ is the adiabatic limit temperature – the temperature the sample would have received under ideal adiabatic conditions

$$T_\infty = \frac{Q}{\rho_1 c_1 e_1 + \rho_2 c_2 e_2} \quad .$$

Here ρ is the density, c the heat capacity and

$$\eta_2 = \frac{e_2}{\sqrt{a_2}} \quad ,$$

with a the thermal diffusivity and

$$d_0 = \cos(\gamma_\mathrm{k}) + PU\cos(U\gamma_\mathrm{k})$$

$$d_e = \frac{\omega_1}{\gamma_\mathrm{k}}\left[Y_1\sin(\gamma_\mathrm{k}) + PUY_2\sin(U\gamma_\mathrm{k})\right] + \frac{\omega_1}{\gamma_\mathrm{k}^2}\left[Y_1\cos(\gamma_\mathrm{k}) + PY_2\cos(U\gamma_\mathrm{k})\right] + \cdots$$
$$\cdots + \frac{Z\omega_1^2}{\gamma_\mathrm{k}^3}\left\{2\left[\sin(\gamma_\mathrm{k}) - P\sin(U\gamma_\mathrm{k})\right] - \gamma_\mathrm{k}\left[\cos(\gamma_\mathrm{k}) - PU\cos(U\gamma_\mathrm{k})\right]\right\}$$

and γ_k is the k^{th} positive root of

$$\sin(\gamma)+P\sin(U\gamma)-\frac{\omega_1}{\gamma}\left\{Y_1\cos(\gamma)+PY_2\cos(U\gamma)+\frac{Z\omega_1}{\gamma}\left[\sin(\gamma)-P\sin(\gamma)\right]\right\}=0$$

The definition of other parameters is written in the Appendix.

2.2 Data reduction

An original software package for experimental ‘flash method’ data processing has been developed [6]. The software currently allows studying of two-layered materials with zero or non-zero thermal contact resistance, respectively, as well as three-layered materials with the ideal thermal contact (uniform - zero thermal contact resistance). The applied theory considers either the ideal adiabatic boundary conditions or heat losses can be taken into account, respectively.

The data reduction consists of a least-squares-fitting the measured temperature rise vs. time evolution. Because we have semi-linear fitting tasks - working expressions linearly depend on the normalization factor - adiabatic limit temperature T_∞, the algorithm described elsewhere [7,8] that shifts the fitting to solving a set of algebraic equations has been reliable implemented.

Results of sensitivity analysis in the case of the presented ideal two-layered composite show that normalized sensitivity to front layer diffusivity a_1 and sensitivity to rear layer diffusivity a_2 vs. time curves are close to being linearly dependent. This indicates that these parameters can’t be estimated simultaneously in a simple flash method experiment [6]. The sensitivity to Biot number Bi and the temperature rise curve that is equal to the sensitivity to the adiabatic limit temperature T_∞ and sensitivity to rear layer diffusivity have different shape. This facts confirm that a simultaneous unique estimation of the adiabatic limit temperature T_∞, of the Biot number Bi and one of the thermal diffusivities a_1 and a_2, respectively, can be performed successfully.

2 Accuracy limits

In order to investigate conditions for the thermal diffusivity determination and accuracy limits the following analyses were performed. We took the two-layered model and calculated various simulated temperature rise vs. time curves. We assumed to have copper and alumina two-layered composite with their relevant (bulk) properties and we considered random heat losses at the level that corresponds to a real laser flash experiment. We calculated simulated temperature rise vs. time curves for various thickness ratios $e_{\mathrm{Cu}}/(e_{\mathrm{Cu}}+e_{\mathrm{alumina}})$ while we considered the total thickness $e=e_{\mathrm{Cu}}+e_{\mathrm{alumina}}$ to have constant. We analyzed how is an estimation of the thermal diffusivity of one layer sensitive to inaccuracy of knowledge the second layer thermal diffusivity. Fig 1 presents results of an estimation of the thermal diffusivity of alumina when the thermal diffusivity of the supplementary component - copper changes in a range of ±1-3%. We see that the accuracy of the determination of the thermal diffusivity of alumina increases when its thickness (e_{alumina}) increases. Similar results can be received for the thermal diffusivity of copper estimation which result varies as a result of deviation of alumina thermal diffusivity. Difference consists in fact, that the accuracy of the determination of the thermal diffusivity of the poorer thermal conductive material (alumina) is much higher that the determination of the higher conductive component. [1].

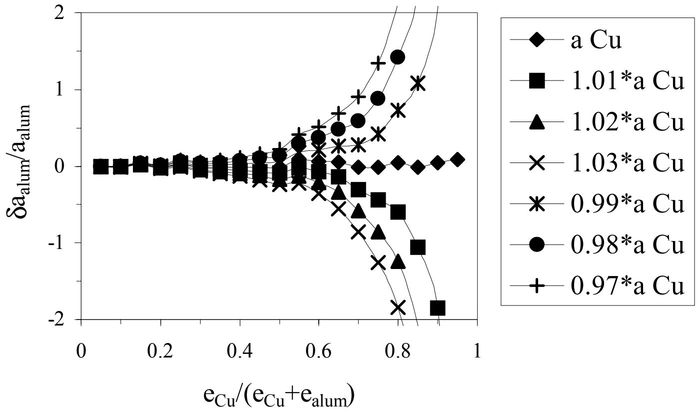


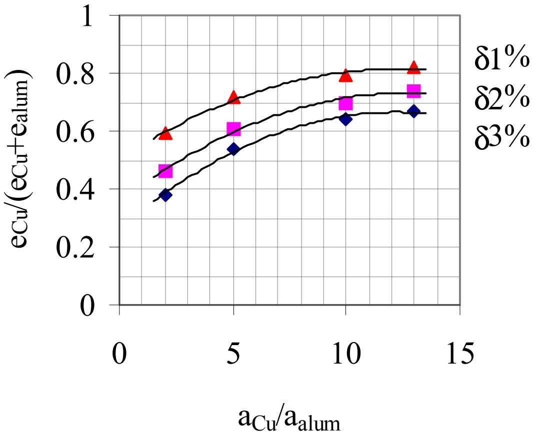
Fig 1 Variation of the thermal diffusivity of alumina vs. thickness ratio caused by deviations of the thermal diffusivity of copper (±3%)

If one performs similar analyses for different component thermal diffusivities ratios the following quantitative relations between the accuracy of the thermal diffusivity estimation and layers thickness ratio can be received. Fig 1 shows that in case the error in the copper thermal diffusivity is less than 3% (2% or 1%, respectively) the accuracy of the thermal diffusivity of alumina estimation better than 1% can be achieved only when the condition eCu/(eCu+ealumina)<0.67 (0.74 or 0.82, respectively) is fulfilled. The choice of the alumina estimation accuracy (1%) we considered is influenced by the fact, that we have not assumed any other disturbing phenomena, as for instance a random noise that can increase the inaccuracy of the measurement to a usual level. Performing similar analyses for different thermal diffusivity ratios, the accuracy limit curves for an estimation of alumina- or copper-like material could be calculated as illustrated in Fig 2 and 3.

Here we note that all the achieved results could be generalized only with a great care. In our analyses, we have not considered variation of the density and the heat capacity. Because of a strong correlation between the thermal diffusivity and the heat capacity - density product in a layered model [8] the achieved accuracy limit curves should be slightly modified. Although any other disturbing phenomena were not taken into account, the achieved results are suitable basis for predetermination of real estimation accuracy limits.

5 Conclusion

The paper shows connection between the inaccuracy of the thermal diffusivities of components on an ideal two-layered composite and their layer-thickness- and thermal-diffusivity-ratios. The proposed approach gives a possibility to relate quantitatively the accuracy of the thermal diffusivity estimation of one layer on a layered composite. Although there were not taken into account other disturbing phenomena (variation of the density and the heat capacity, noise, etc.) the achieved results give suitable basis for predetermination of real estimation accuracy limits.



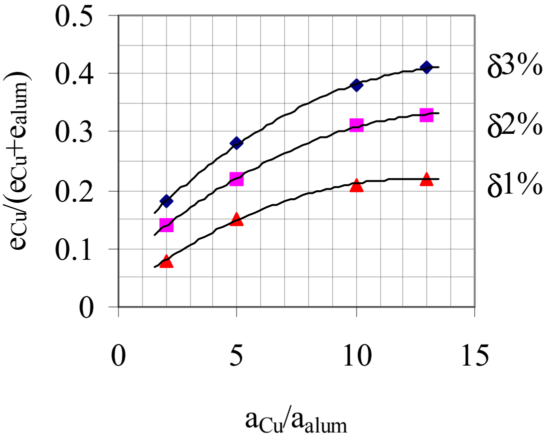


Fig 2 Accuracy limit curves of the thermal diffusivity of alumina estimation as a function of the thermal diffusivity ratios and thermal diffusivity of copper variation (±1-3%)

Fig 3 Accuracy limit curves of the thermal diffusivity of copper estimation as a function of the thermal diffusivity ratios and thermal diffusivity of alumina variation (±1-3%)

Appendix

Parameters in the text are defined as follows

$H_i = A\, \rho_i c_i e_i$

$$\eta_i = \frac{e_i}{\sqrt{a_i}}$$

where *A* is the cross-sectional area of the sample and

$H_{i/j} = H_i/H_j$

$\eta_{2/1} = \eta_2/\eta_1$

$X_1 = H_{1/2}\,\eta_{2/1} + 1$

$X_2 = H_{1/2}\,\eta_{2/1} - 1$

$\omega_1 = \eta_{1/2} + 1$

$\omega_2 = \eta_{1/2} - 1$

$$P = \frac{X_2}{X_1}$$

$$U = \frac{\omega_2}{\omega_1}$$

$$Bi_i = \frac{h_i e_i}{a_i c_i \rho_i}$$

$$Bi_{1/2} = \frac{Bi_1}{Bi_2}$$

$$Y_1 = Bi_2\left(\frac{Bi_{1/2}}{\eta_{1/2}} + 1\right)$$

$$Y_2 = Bi_2\left(\frac{Bi_{1/2}}{\eta_{1/2}} - 1\right)$$

$$Z = Bi_2^2\,\frac{Bi_{1/2}}{\eta_{1/2}}$$

Acknowledgement

Authors wish to thank the Slovak Science Grant Agency and the Hertha Firnberg Foundation of the ARCS for their financial support.

References

[1] Hohenauer W, Vozár L, An Estimation of Thermophysical Properties of Layered Materials using the Laser Flash Method, *High Temp. High Press.* (accepted)

[2] Parker W J, Jenkins R J, Butler C P and Abbott G L, 1961 Flash Method of Determining Thermal Diffusivity, Heat Capacity and Thermal Conductivity, *J. Appl. Phys.* **32**, 1679-1684

[3] Balageas D L, 1989 Thermal Diffusivity Measurement by Pulsed Methods, *High Temp. - High Press.* **21**, 85-96

[4] Maglić K D, Taylor R E, 1992 The Apparatus for Thermal Diffusivity Measurement by the Laser Pulse Method, in *Compendium of Thermophysical Property Measurement Methods Vol.2 Recommended Measurement Techniques and Practices,* Eds K D Maglić, A Cezairliyan, V E Peletsky (New York: Plenum Publishing Corp.) p.281

[5] Sweet J N, 1989 in *Thermal Conductivity 20* Ed D P H Hasselman, J R Thomas, Jr. (New-York: Plenum) pp. 287-304

[6] Vozár L, 1999 Measurement of the Thermal Diffusivity and the Contact Thermal Resistance of Two- and Three-Layered Composite Materials Using the Flash Method, Progress report ARC Seibersdorf, CPU Nitra

[7] Gembarovič, J., Vozár, L., Majerník, V., 1990 Using the Least Squares Method for Data Reduction in the Flash Method, *Int. J. Heat Mass Transfer* **33** 1563-1565

[8] Vozár, L., Šrámková, T., 1997 Two data Reduction Methods for Evaluation of Thermal Diffusivity from Step Heating Measurements, *Int. J. Heat Mass Transfer* **40** 1647-1656

[9] Koski J A, 1985 in *Thermal Conductivity 18*, Ed T Ashworth, D R Smith (New-York: Plenum) pp. 525-536

STEADY-STATE THERMAL CONDUCTIVITY DETERMINATION OF NON-METALLIC MATERIALS AT CONSTANT SURFACE HEAT SOURCE

Peter Matiašovský, Oľga Koronthályová

Department of Building Physics, Institute of Construction and Architecture, Slovak Academy of Sciences, Dúbravská cesta 9, 842 20 Bratislava
usarmat@savba.sk, usarkoro@savba.sk

Abstract

In determination of thermal conductivity generally the important problem is to realize satisfactorily defined conditions on the boundaries of a specimen. A possible solution is to realize steady-state boundary conditions of 3rd kind defining heat transfer coefficient between specimen and measuring device. Determination of thermal conductivity is then possible from analytical solution of Laplace equation for the case of three-dimensional body heated by square surface heat source. The thermal conductivity is calculated from measured data on heat flow and temperature difference.

Key words: building materials, thermal conductivity, stationary measuring methods

1 Introduction

Porous building materials are characterised by densities in the range 20 - 2 500 kg/m^3 and thermal conductivities in the range 0.03 - 1.5 W/m.K. Commonly used methods of thermal conductivity determination for porous building materials are plate methods and probe methods. Each type of the methods has its own merits and insufficiencies. Plate methods require relatively long measuring times, large specimens and ideal thermal contact mainly in the case of thin highly conductive specimens. Probe methods are quick but they are not able to evaluate heterogeneous materials and they require ideal thermal contact between specimen and probe too. For practical use a simple and quick measuring method should have following properties: satisfactorily large contact area, no limitations regarding the specimen dimensions and shape, no problems with realization of thermal contact. A possible solution is method based on steady state heating of specimen surface by constant heat flow with defined square thermal contact area and heat transfer coefficient. Measuring device providing these conditions have been developed and tested.

2 Theory

We consider infinite homogeneous plate of thickness H. At a surface of the plate a constant heat source of square area is placed. Temperature of environment is t_a (Fig. 1). The thermal conductivity of the plate is detemined from analytical steady-state solution of the Laplace equation. The mathematical formulation of the problem is given by following equations:
Laplace equation:

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2} = 0 \tag{1}$$

Temperature of the contact face of square heat source:

$$t = t_i \quad \text{for} \quad 0 < |x| < \frac{L}{2} \quad \text{and} \quad 0 < |y| \le \frac{B}{2} \tag{2}$$

Temperature of the environment:

$$t = t_a \quad \text{for} \quad \frac{L}{2} < |x| \le \infty \quad \text{and} \quad \frac{B}{2} < |y| \le \infty \tag{3}$$

Temperature of the opposite non-heated surface:

$$t = t_e \tag{4}$$

Boundary conditions of 2nd kind at the surfaces perpendicular to the heated one are provided by sufficiently large x_1, y_1. If the dimensions of the plate are higher than x_1, y_1 the problem can be applied for the body of finite dimensions: rectangular parallellepiped, finite cylinder, etc.

$$\frac{\partial t}{\partial n} = 0 \tag{5}$$

Boundary conditions of 3rd kind at the surface with the heat source:

$$\alpha \cdot [t(x,y) - F(x,y,0)] = -\lambda \cdot \left| \frac{\partial F(x,y,0)}{\partial z} \right| \tag{6}$$

The solution of the steady state problem for heat flow into the plate [1] is:

$$Q = \lambda \cdot L \cdot \frac{t_a - t_e}{H + \frac{\lambda}{\alpha}} + \lambda \cdot L \cdot (t_i - t_a) \cdot \mathrm{FUNK}(\rho, \eta, \tau) \tag{7}$$

$$\mathrm{FUNK}(\rho,\eta,\tau) = \frac{8}{\pi^3 \mu} \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} \frac{\sin(\pi.\mu.n)}{n^2} \frac{\sin(\pi.\mu.i)}{i^2} \cdot \cdot \frac{1}{2 \cdot \tau \cdot \pi \cdot \mu + \frac{1 - e^{-4 \cdot \pi \cdot \mu \cdot \rho \cdot w}}{w \cdot \left(1 + e^{-4 \cdot \pi \cdot \mu \cdot \rho \cdot w}\right)}} \tag{8}$$

where:

$$w = n + \frac{i}{\eta} \tag{9}$$

$$\eta = \frac{L}{B} \tag{10}$$

L and B are dimensions of the square heat source [m]

$$\mu = \frac{L}{2 \cdot x_1} = \frac{B}{2 \cdot y_1} \tag{11}$$

x_1, y_1 are dimensions of the sample in the plane of the heat source [m]

$$\rho = \frac{H}{B} \tag{12}$$

H is thickness of the sample [m]

$$\tau = \frac{\lambda}{\alpha \cdot B} \tag{13}$$

λ is thermal conductivity of the sample [W/m.K], α is heat transfer coefficient between the source and the sample and between the sample and its surroundings [W/m^2.K]. The defined constant value of heat transfer coefficient over the whole exposed surface area is provided by construction of the measuring device.

Thermal conductivity of the specimen is calculated as a root of transcendental equation (7) considering that the measured heat flow Q and temperature difference t_i - t_a are parameters.

3 Measuring device

Schematic description of the measuring set is in Fig. 1. The measuring device consists of three components: heat source plate, temperature sensor plate and heat flow meter composed in series in the direction of heat flow. Dimensions of the device are of 0.03x0.03x0.0082 m and it operates simultaneously as heating and measuring unit.

The standard part of the measuring set is aluminium guard frame. During the measurement the guard frame is affixed to the heated specimen surface in order to provide the same heat transfer coefficient value on the whole specimen's exposed surface as the contact heat transfer coefficient between the contact face of the measuring unit and the specimen.

The problem of edge heat loss along the perimeter of the measuring unit is solved by construction solution, which provides that the heat loss is proportional to heating input. The heat loss of a measuring device as well as the heat transfer coefficient value can be determined from the etalon measurements. The calibrations have been carried out from experiments with etalons of expanded polystyrene, polymethylmethacrylat, aerated autoclaved concrete, clay brick and concrete. The edge heat loss as a function of total heating is shown in Fig. 2. The heat transfer coefficient has value ca 130 W/m^2.K in the case of porous materials as a result of non-ideal contact.

The sample can have practically arbitrary shape and dimensions: semiinfinite body, infinite plate, plate of finite dimensions, parallelepiped etc. The range of possible determined thermal conductivities is 0.025 - 1.5 W/m.K, practical temperature interval 50 do 85 °C.

4 Conclusion

Simple steady-state method for determination of thermal conductivity during specimen heating by constant square surface heat source is applicable for thermal conductivities between 0.04 - 1.5 W/m.K.

The specific feature of the method is possibility to use defined heat transfer on the contacts between the specimen and heat source and between the specimen and its environment instead the realization of an ideal thermal contact.

Heat loss at the edge of contact plane between the specimen and the device is calibrated.

The size of used specimens is practically unlimited and the measuring device can be utilised as for the plane structures in situ as for small specimens in laboratory conditions.

The time of one measurement is ca 20 minutes.

Acknowledgement

The authors are grateful to VEGA (Grant No. 2/5105/99) for the supporting of this work.

References

[1] Krischer O, 1934 Die Wärmeaufnahme der Grundflächen nicht unterkellerter Räume. *Gesundheitsingenieur* **39**, 513-521

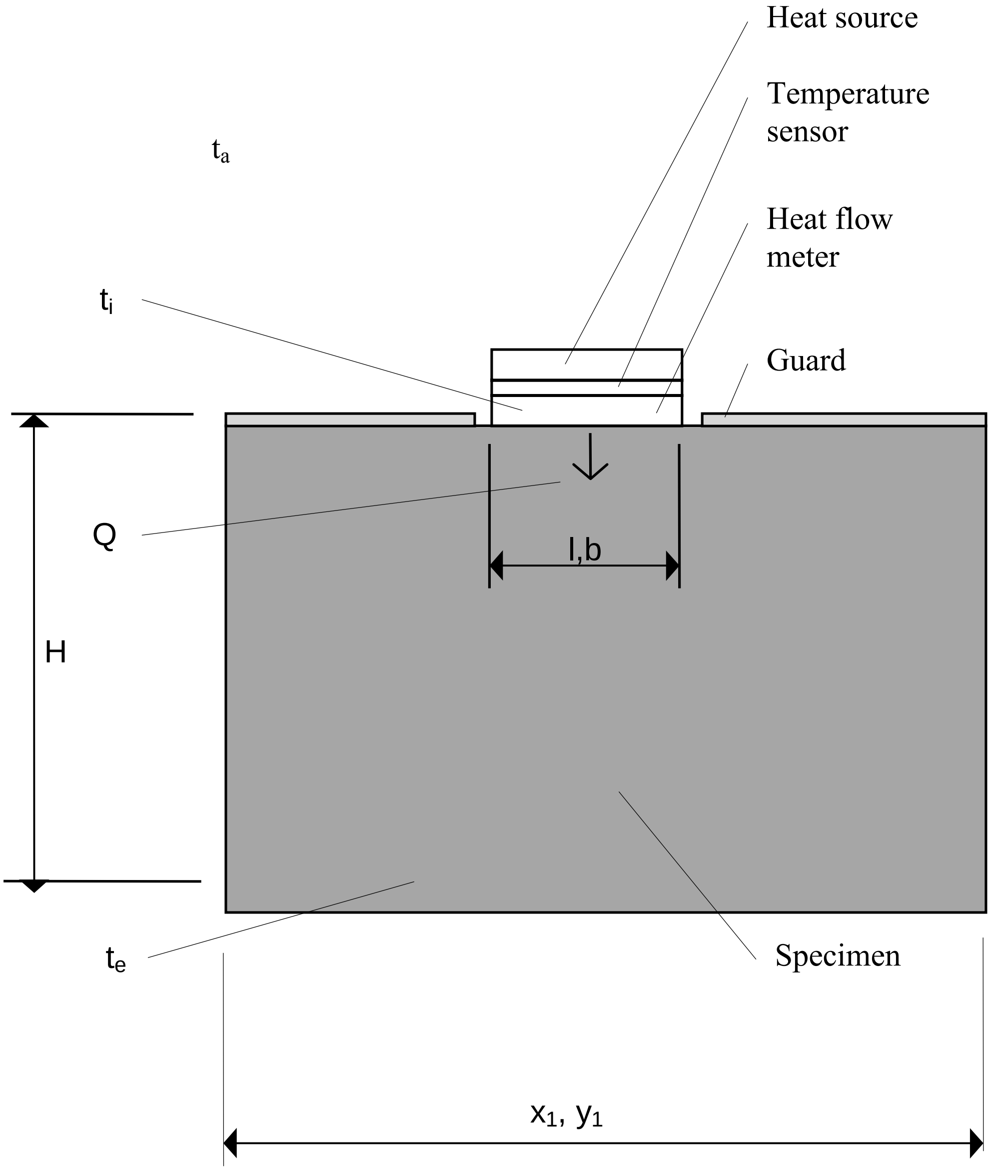


Fig. 1 Schematic description of the measuring device

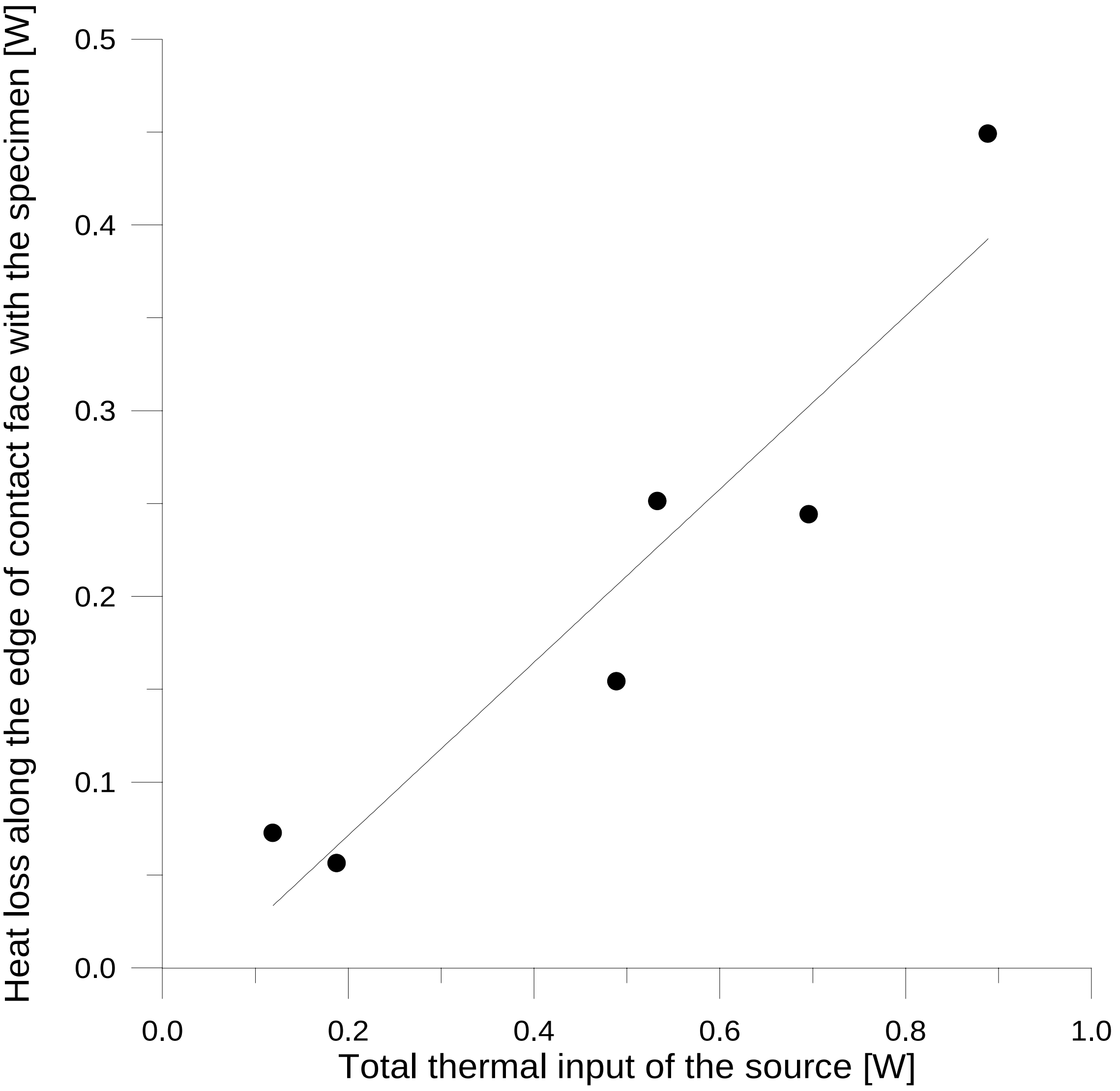


Fig. 2 Dependence of heat loss along the edge of the contact face with the specimen on the total heat source input

METHODOLOGY OF STEP – WISE METHOD FOR MEASURING THERMOPHYSICAL PARAMETERS OF MATERIALS

Ľudovít Kubičár, Vlastimil Boháč

Institute of Physics, Slovak Academy of Sciences, Dúbravská cesta 9, 842 28
Bratislava, Slovakia
Email: fyzikubo@nic.savba.sk, fyzibohc@nic.savba.sk

Abstract. A version of transient method for measuring specific heat, thermal diffusivity and thermal conductivity is presented. The dynamic temperature field is generated by the passage of the electric current through a plane electrical resistance made of thin metallic foil. The heat is produced in the form of step – wise function. Theory of the method, its experimental arrangement and the measuring regime considering sensitivity coefficients are presented. Experimental data obtained on PERSPEX are inter-compared with recommended and published data. Data of thermal conductivity agrees within ± 0.7% while specific heat is shifted in average within -5.2% and thermal diffusivity within +3.5% when measurements were realized in vacuum

Keywords: transient method, step-wise method, specific heat, thermal diffusivity, thermal conductivity

1 Introduction

Various methods are used for measurement of specific heat c_p, thermal diffusivity a, and thermal conductivity λ. Most of them give one parameter, only. Maglic, Cezayirliyan and Peletsky [1] has given a review of the recommended measuring techniques. Kubicar and Bohac has given an overview of the transient methods [2].

Transient methods are based on the generation of a dynamic temperature field inside the specimen. The measuring process can be described as follows: The temperature of the specimen is stabilized and uniform. Then a small disturbance in the form of a pulse of heat or a heat flux in the form of a step-wise function is applied to the specimen. From the temperature response to this small disturbance the thermophysical parameters can be calculated according to the model used.

The presented paper is focused on the step-wise transient method that belongs to the class of transient ones. Theory and the experimental technique of the step – wise transient will be presented. Considerations are concentrated to the measuring regime, data analysis and inter – comparison measurements made on PERSPEX (polymethylmetacrylate) using published and recommended data.

2 Theory

The principle of the step – wise transient is shown in Fig. 1. The specimen is cut into three pieces. The dynamic temperature field is generated by the passage of the electric current through a plane electrical resistance made of thin metallic foil.

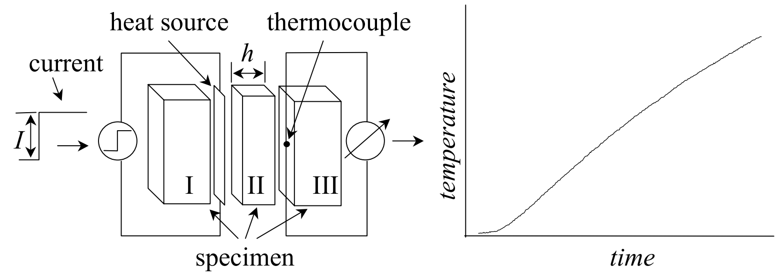


Fig. 1 Principle of the step–wise transient method

The temperature response is measured by a thermocouple. Both the thermocouple and the heat source are placed between the cut surfaces of the specimen.

The model of the method is characterized by the temperature function [3]

$$T_i(F,h) = \frac{qh}{c\rho a}\left\{\frac{1}{2}\sqrt{\frac{F}{\pi}}\exp(-\frac{1}{4F}) - \frac{1}{2}erfc(\frac{1}{2\sqrt{F}})\right\} \tag{1}$$

where $F = at/h^2$, ρ is density, $q_0 = RI^2$ is a heat supplied from the unit area of the heat source into the specimen, R is electrical resistance of the unit area of the heat source, and other symbols are shown in Fig. 2.

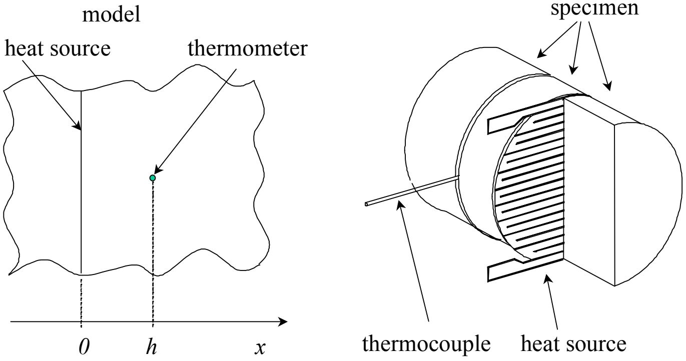


Fig. 2 Model and the experimental set when using step – wise transient.
A part of the specimen is cut to show the form of the heat source.

Solution (1) is given for ideal model. Real experimental set up is shown at the right side while ideal model on the left side of the Fig. 2. Clear differences exist between the model and the experimental set up. The heat is produced in a metallic foil having a real thickness and different thermophysical properties in comparison to the specimen while ideal model assumes that heat source of zero thickness has thermophysical properties identical with those of the specimen. A contact thermal resistance exists between the heat source and the specimen. Similar situation can be found at the contact where the thermocouple is placed. Moreover, model assumes non-limited body while real experiment works with limited specimen. Thus, specimen surfaces might influence the measurement process. Therefore the theoretical analysis has to be performed in which the initial and boundary conditions would be formulated in a more realistic way. The analysis of such type of solutions gives criteria when ideal model, in fact the function (1), can be used.

The thermophysical parameters can be found using the temperature function (1) over the temperature response by appropriate fitting technique. Two problems exist, namely how long should be the temperature response scanned and what time period for fitting should be used. The sensitivity coefficients give measuring time (time during which the temperature response is scanned). Moreover sensitivity coefficients and the analysis of the correlation give time window in which the evaluation technique can be applied over the temperature response.

The sensitivity coefficient β_p is given by [4]

$$\beta_p = p\frac{\partial T_i(F,h)}{\partial p} \tag{2}$$

where p is parameter to be analyzed and $T_i(F,h)$ is the temperature function (1).

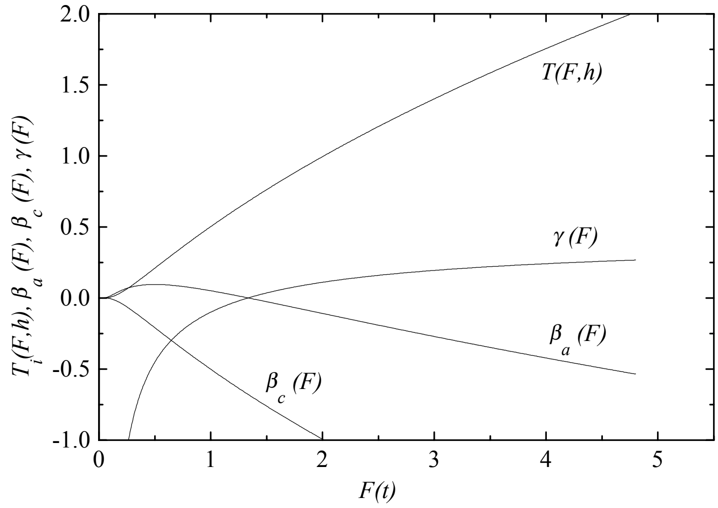


Fig. 3 Correlation $\gamma(t) = \beta_a / \beta_c$ and sensitivity coefficients β_a and β_c as functions of the Fourier number $F = at/h^2$. Curves are calculated for PERSPEX .

The correlation was analysed in two ways namely by the function $\gamma(t) = \beta_a(t)/\beta_c(t)$ [5] and by difference analysis. Fig. 3 shows the temperature response, sensitivity coefficients β_a, β_c and the result of the correlation analysis given by function $\gamma(t)$. Theoretical points using function (6) and parameters corresponding to PERSPEX, i.e. thermal diffusivity $a = 0.12\ 10^{-6}\ \mathrm{m}^2\ \mathrm{sec}^{-1}$, density $\rho = 1184\ \mathrm{kg\ m}^{-3}$, specific heat $c_p = 1254\ \mathrm{J\ kg}^{-1}\ \mathrm{K}^{-1}$ and specimen thickness $h = 0.005$ m were calculated. Four digits were used, only to model a real experiment. Sensitivity coefficient β_a has a maximum at

$F = 0.5$ while β_c is changed systematically. For high correlation, the function $\gamma(t)$ is approaching $\gamma(t) \rightarrow$ const. It is a region for $F > 2$. Difference analysis is based on the fitting of the function (1) over the theoretical points. The points were calculated every 0.2 sec up to 1000 sec. A strobe (time interval) of 10 sec was chosen in which the fitting procedure was applied over a part of the theoretical response using, again, function (1). The strobe was consecutively shifted over theoretical temperature response starting from 0.2 sec (the first used strobe 0.2 – 10 sec) up to 500 sec (the last used strobe 490 – 500 sec) in steps 0.2 sec. Fitted values of specific heat and thermal diffusivity are shown in figure 4a, b as functions of the Fourier number $\overline{F}$ that corresponds to the mean time of the strobe. The fitting procedure works with T and t variables (relation (1)) while results shown in figure 4a,b are shown in variables T and $\overline{F}$. This causes multiple values when correlation exists. Data stability is poor outside the interval $0.9 < \overline{F} < 1.7$ where high correlation exists even the function (1) is fitted over its theoretical points.

The temperature response is scanned up to the moment when the correlation of the sensitivity coefficients starts to be high, i.e. up to $\overline{F} \sim 2$. This criterion gives the measuring time. Low correlation represents situation when the sensitivity coefficients are linearly independent. It is the interval $0.7 < \overline{F} < 1.7$. Time window in which the fitting procedure can be applied is $0.7 < F < 1.7$ considering results of difference analysis.

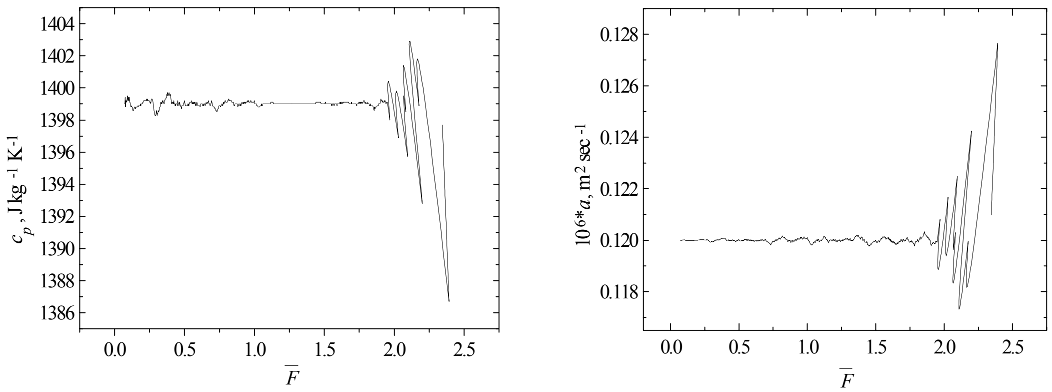


Fig. 4(a,b) Specific heat and thermal diffusivity as a function of the middle time of the strobe. Strobe was chosen to be 10 sec ($\Delta F = 0.12$).

3 Experiment

The diameter of the specimen is 30 mm. The specimen is cut into three pieces (figure 2). The heat source is made of nickel foil of thickness 20 μm and diameter 30 mm. The electrical resistance of the heat source was ~ 2 Ω. A thermometer, made of thermocouple Chromel – Alumel of thickness of 50 μm, is placed apart of the heat source in a distance of 5 mm. A heat sink paste (Midland Silicones Barry Glamorgan) is used to improve the thermal contact between the individual parts of the specimen set. A Thermophysical Transient Tester Model RT 1.02 (Institute of Physics SAS) was used for measurement of thermophysical parameters.

4 Results and discussion

A test of correlation was made to find experimentally the measuring time and the time window suitable for evaluation in the following way: The temperature response was scanned over 1000 sec (measuring time up to $F = 10$). Time difference between two scans was 3.3 sec. A strobe of 60 sec ($\Delta F = 0.3$) was chosen in which the specific heat and thermal diffusivity was found by fitting procedure. Smaller strobe causes data scattering (figure 5a, b).

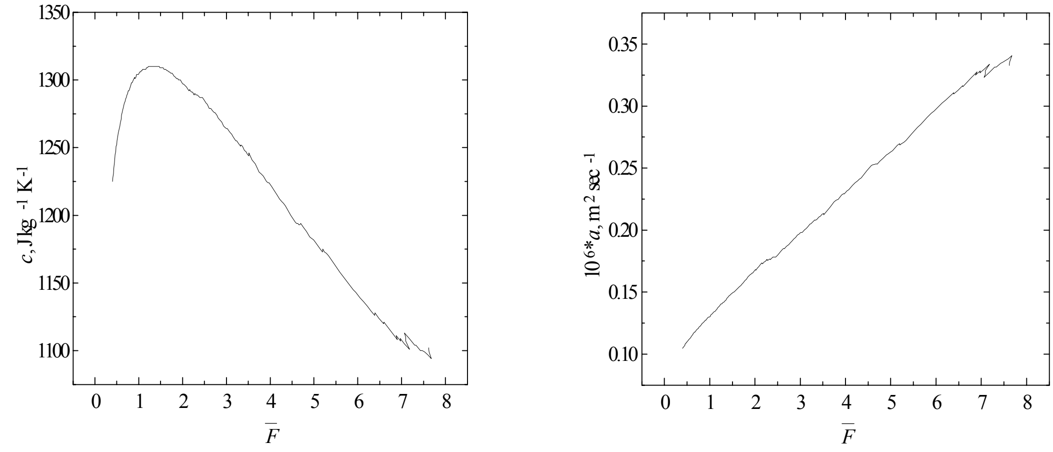


Fig. 5(a, b) Specific heat (a) and thermal diffusivity (b) as a function of the middle time of the strobe. Strobe is 60 sec ($\Delta F = 0.3$).

The strobe was applied over the whole temperature response by shifting it in steps from the beginning to the end. One step represents a shift of 3.3 sec. While values of the specific heat are stabile in $0.8 < \overline{F} < 1.4$ thermal diffusivity values change over the whole measuring time. An average value of thermal diffusivity will be obtained when using time window $0.8 < F < 1.4$.

The measurements were performed in the measuring cycles. The measuring time (time period during which the temperature field is generated and the temperature response is scanned) t_{meas} and the length of the measuring cycle t_{cycle} was chosen $t_{meas} = 400\ sec$ ($F = 1.92$) and $t_{cycle} = 1\ hour$, respectively. A fitting procedure was applied in the time window 166 – 291 sec to find values of the specific heat and thermal diffusivity.

Averaged values and standard deviations are plotted in figure 6 as a function of the sequence number namely in the upper part for thermal diffusivity, in the middle for specific heat and at the bottom for thermal conductivity. Averaging was made upon data that were obtained at the same heat output of the heat source. Lines represent published values for specific heat and thermal diffusivity [6] and recommended data for thermal conductivity given by NPL [7].

Differences in values of transport parameters (thermal conductivity and diffusivity) measured in air and in vacuum was found. Clearly, heat loss from the specimen surface influences the measuring process in air for longer time. The criterion of ideal model was fulfilled for vacuum, only when heat losses from the specimen surface was suppressed.

Deviations exist in values of specific heat within – 5.2% and thermal diffusivity within +3.5%. A more detailed study has to be done to find the reason of data shift.

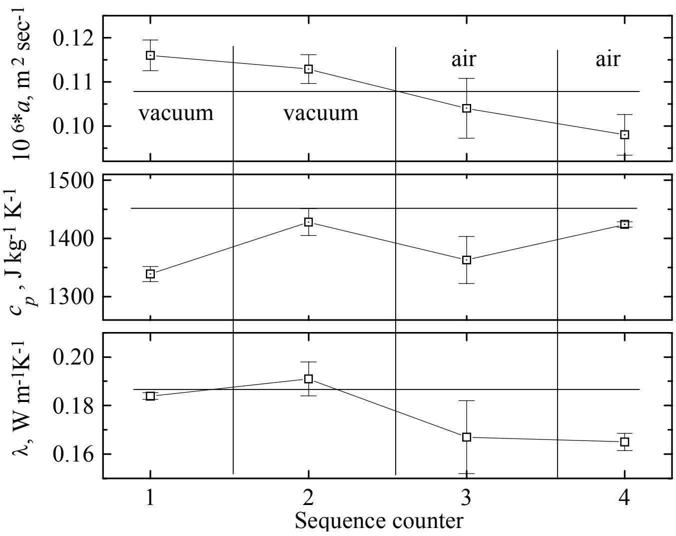


Fig. 6 Thermal diffusivity, specific heat and thermal conductivity of PERSPEX as a function of the measurement sequence. Recommended value for thermal conductivity (NPL [7]) and published values for specific heat and thermal diffusivity [6] are marked by horizontal lines. Specimen temperature $T = 25^{o}$C.

5 Conclusions

Data of thermophysical parameters of PERSPEX, namely specific heat, thermal diffusivity and thermal conductivity were obtained by step – wise transient method. Deviations exist in values of specific heat within – 5.2% and thermal diffusivity within +3.5%. A more detailed study has to be done to find the reason of data shift.

Acknowledgements

The authors are grateful to the Thermophysical Properties Section at the National Physical Laboratory for providing specimens PERSPEX, and their comments on early draft of this paper. This work was supported by grant agency VEGA under Nr. 2/5084/98.

References

[1] Maglić K D, Cezairliyan A and Peletsky V E, 1992 *Compendium of Thermophysical Property Measurement Methods Vol 2 Recommended Measurement Techniques and Practices* (New York: Plenum)

[2] Kubičár L, Boháč V, 1997 Review of Several Dynamic Methods of Measuring Thermophysical Parameters, in *Thermal Conductivity 24* (Pittsburgh: Technomic Publication Company, Inc.) p.135
[3] Carslaw H S, Jaeger J C, 1959 *Conduction of Heat in Solids* (London: Clarendom Press)
[4] Beck J V, Arnold K J, 1977 *Parameter Estimation in Engineering and Sciences* (New York, London, Sidney, Toronto: John Willey)
[5] Raynaund M, 1999 Strategy for Experimental Design and the Estimation of Parameters, *High Temp. High Press.* **31**, 1105–1119
[6] Vasiliev L L, Tanajev S A, 1971 *Teplofiziceskije svojstva poristich materialov* (Minsk: Nauka i Technika)
[7] Salmon D R, 1999 personal communication National Physical Laboratory Teddington TW11 0LW UK

MATRIX SOLUTION OF THE HEAT CONDUCTION IN MULTI-LAYERED STRUCTURES IN GENERAL CASE

Aba Teleki

Department of Physics, Faculty of Natural Sciences, Constantine the Philosopher University, Tr. A. Hlinku 1, SK-94974 Nitra, Slovakia
Email: ateleki@ukf.sk

Abstract

It is useful to solve the thermophysical properties of layered materials by the method of so-called matrix–form formalism ([1], [2]). This one is used for two- and free-layered system with one source on the form of Dirac´s function to modeling the experimental data (see [1]). The solution for *n* sandwich multi-layer sytem was given in the case of one dimensional flow. No heat loss from the sample surface, no thermal contact resistance between layers were assumed there. Further homogenous layers and constant thermophysical properties of each layer (see [2]) were pressumed. We studied the more general case here. We show the multi-pole solution of temperature conduction in the case of one dimensional heat flow with heat loss from the sample surface, assuming thermal contact resistance between layers, heat input on the surfaces between every layer and in the body of the slabs also. Homogoneous, isotropic layers with constant themophysical properties of each layers were assumed also.

Key words: layered structures, multipole solution, general solution, Laplace transform

1 Introduction

The interesting features of composite materials require studies of their heat conduction properties. The native form of these studies is the modeling of heat conduction in multi-layered structures. One of the more convenient methods of modeling is using matrix-form of the solutions (see [1], [2]). The system of equations of heat conduction is presented in a series of matrices in this case. There are clear and strong relation between the physical situation in every slab and the corresponding subset of this matrix series. A similar method used in [2] where they assume no heat sources in slabs, and trivial initial conditions of the temperature in $t=0$. The only one heat source was placed on the outer surface of the first slab.

The main goal of this article is to show the compact form and the structural correspondence between slabs (their internal or boundary properties) and matrix–form solutions in more general case. Therefore we are not looking for final solution (by probing to invert the Laplace transform of the solutions).

It is to be noted that the matrix-form method may be simply modified for separable variables also (without using the Laplace transformation).

We consider a multi-layer composite material shown in Figure 1. which one are infinite in two directions. The boundary indexed *L* borders the multi-layer system from left and similarly the boundary indexed R borders the multi-layer system from right (it is convenient to substitute 0 for *L* and *n*+1 for *R* respectively).

In this article α_j denotes the thermal diffusivity, k_j the thermal conductivity, h_j the heat-transfer coefficient of the layer indexed *j*. We assume the coefficients to be constants. Temperature is $T_j(x,t)$, $g_j(x,t)$ describes heat-source and $F_j(x)$ is the temperature profile at $t=0$. We suppose the existence of heat-sources on the boundary between the *j*-th and (*j*+1)-th layer, represent by $W_j(t)$. The coordinate of the boundary plane between the *j*-th and (*j*+1)-th slab is denoted by x_j. We assume the material of each slab to be homogenous and isotropic. We introduce the index set $S_n=\{1,\ldots,n\}$..

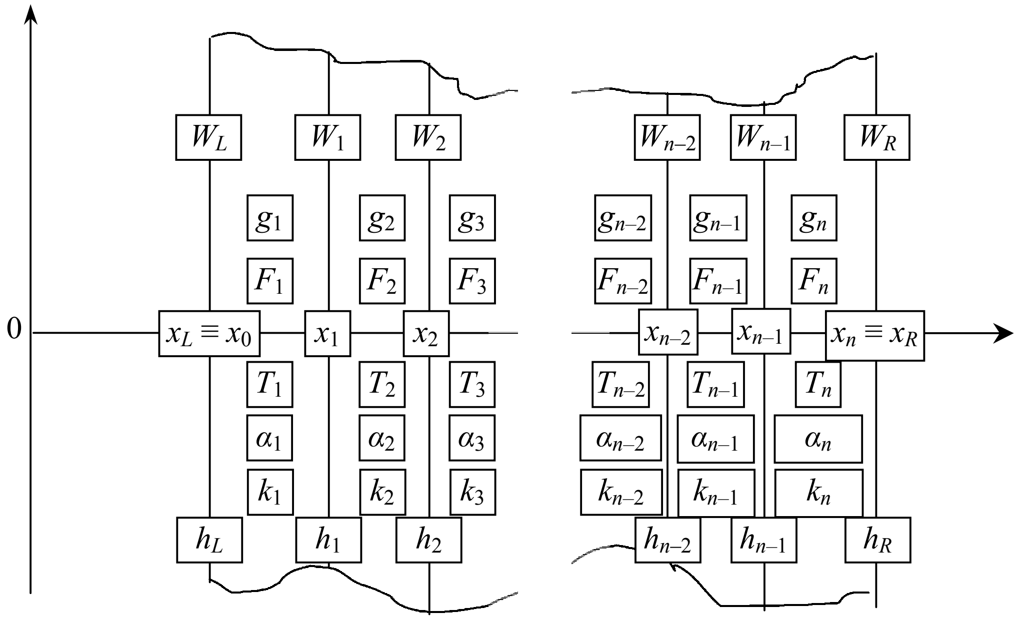


Fig. 1 Multi-layer composite region

The partial differential equation for heat conduction is given in the form:

$$\partial_{xx}T_j(x,t)-\alpha_j^{-1}\partial_t T_j(x,t)+k_j^{-1}g_j(x,t)=0 \quad \texttt{for } t>0, j\in S_n \tag{1}$$

and the boundary conditions are

$$\left[k_j\partial_x T_j(x,t)+h_{j+1}\left(T_j(x,t)-T_{j+1}(x,t)\right)\right]_{x=x_j}=W_j(t) \quad \texttt{for } t>0, j\in S_{n-1}, \tag{2}$$

$$\left[k_j\partial_x T_j(x,t)-k_{j+1}\partial_x T_{j+1}(x,t)\right]_{x=x_j}=0 \quad \texttt{for } t>0, j\in S_{n-1}, \tag{3}$$

$$T_j(x,0)=F_j(x). \tag{4}$$

For the left and right boundaries we have

$$\left[k_L \partial_x T_1(x,t) - h_L T_1(x,t)\right]_{x=x_L} = W_L(t), \tag{5}$$

$$\left[k_R \partial_x T_n(x,t) + h_R T_n(x,t)\right]_{x=x_R} = W_R(t). \tag{6}$$

The Laplace transform L_s of these equations gives the following differential equations

$$\begin{gathered}\partial_{xx}\theta_j(x,s) - \alpha_j^{-1} s\theta_j(x,s) + \Gamma_j(x,s) = 0 \quad \text{for } j \in S_n \quad \text{where} \\ \theta_j(x,s) \equiv \mathrm{L}_s\{T_j(x,t)\}, \Gamma_j(x,s) \equiv F_j(x) + k_j^{-1}\gamma_j(x,s) \quad \text{and } \gamma_j(x,s) \equiv \mathrm{L}_s\{g_j(x,t)\},\end{gathered} \tag{7}$$

$$\begin{gathered}\left[k_j \partial_x \theta_j(x,s) + h_{j+1}\left(\theta_j(x,s) - \theta_{j+1}(x,s)\right)\right]_{x=x_j} = \omega_j(s) \quad \text{for } j \in S_{n-1} \quad \text{where} \\ \omega_j(s) \equiv \mathrm{L}_s\{W_j(t)\},\end{gathered} \tag{8}$$

$$\left[k_j \partial_x \theta_j(x,s) - k_{j+1}\partial_x \theta_{j+1}(x,s)\right]_{x=x_j} = 0 \quad \text{for } j \in S_{n-1}, \tag{9}$$

$$\left[k_L \partial_x \theta_1(x,s) - h_L \theta_1(x,s)\right]_{x=x_L} = \omega_L(s) \quad \text{where } \omega_L(s) \equiv \mathrm{L}_s\{W_L(t)\} \quad \text{and} \tag{10}$$

$$\left[k_R \partial_x \theta_n(x,s) + h_R \theta_n(x,s)\right]_{x=x_R} = \omega_R(s) \quad \text{where } \omega_R(s) \equiv L_s\{W_R(t)\}. \tag{11}$$

We can write the general solutions of the system of differential equations (7) in the form

$$\theta_j(x,s) = A_j^- \theta_j^-(x,s) + A_j^+ \theta_j^+(x,s) + \theta_j^p(x,s) \quad j \in S_n \tag{12}$$

where $\theta_j^+(x,s)$ and $\theta_j^+(x,s)$ are independent solutions of the homogenous equations obtained from (7). Functions $\theta_j^p(x,s)$ are partial solutions of the inhomogeneous system of equations in the interval (x_{j-1}, x_j). One of the forms of independent homogenous solutions is:

$$\theta_j^-(x,s) = \sinh\left(\sqrt{\frac{s}{\alpha_j}}\, x\right) \quad \text{and} \quad \theta_j^+(x,s) = \cosh\left(\sqrt{\frac{s}{\alpha_j}}\, x\right) \quad j \in S_n. \tag{13}$$

We keep the general form of the solutions for farther calculations:

2 The matrix-form solutions

We can express the system of differential equations (8), (9), (10) and (11) using (12), as a system of linear equations for the set of $2n$ unknown variables $\left\{A_j^-, A_j^+\right\}_{j \in S_n}$ as follows

$$\mathbf{M.A} = \mathbf{Z}\,. \tag{14}$$

The matrix **M** is a band-diagonal square matrix with block elements

$$\mathtt{M} = \begin{pmatrix} \mathbf{D}_1 & -\mathbf{M}_1 & \mathbf{0} & \cdots & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_2 & -\mathbf{M}_2 & \mathbf{0} & & \vdots \\ \vdots & \mathbf{0} & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \mathbf{D}_{n-2} & -\mathbf{M}_{n-2} & \mathbf{0} \\ \vdots & & & \mathbf{0} & \mathbf{D}_{n-1} & -\mathbf{M}_{n-1} \\ \mathbf{0} & \cdots & & \cdots & \mathbf{0} & \mathbf{D}_n \end{pmatrix} \quad \text{and} \tag{15}$$

and the matrices **A** and **Z** are column matrices defined as

$$\mathtt{A} = \begin{pmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \vdots \\ \mathbf{A}_n \end{pmatrix}, \quad \mathbf{A}_j = \begin{pmatrix} A_j^- \\ A_j^+ \end{pmatrix} \quad \text{and} \quad Z = \begin{pmatrix} \mathbf{Z}_1 \\ \mathbf{Z}_2 \\ \vdots \\ \mathbf{Z}_n \end{pmatrix}, \quad \mathbf{Z}_j = \begin{pmatrix} z_1^{(j)} \\ z_2^{(j)} \end{pmatrix}, j \in S_n . \tag{16}$$

The block elements $\mathbf{D}_j$ and $\mathbf{M}_j$ are 2×2 matrices defined as follows from (12) and (8)–(11)

$$\mathbf{D}_j \equiv \begin{pmatrix} \left[k_j \partial_x \theta_j^-(x,s) + h_{j+1}\theta_j^-(x,s)\right]_{x=x_j} & \left[k_j \partial_x \theta_j^+(x,s) + h_{j+1}\theta_j^+(x,s)\right]_{x=x_j} \\ \left[k_j \partial_x \theta_j^-(x,s)\right]_{x=x_j} & \left[k_j \partial_x \theta_j^+(x,s)\right]_{x=x_j} \end{pmatrix} \quad \text{for} j \in S_{n-1}, \tag{17}$$

$$\mathbf{M}_j \equiv \begin{pmatrix} \left[h_{j+1}\theta_j^-(x,s)\right]_{x=x_j} & \left[h_{j+1}\theta_{j+1}^+(x,s)\right]_{x=x_j} \\ \left[k_{j+1}\partial_x \theta_{j+1}^-(x,s)\right]_{x=x_j} & \left[k_{j+1}\partial_x \theta_{j+1}^+(x,s)\right]_{x=x_j} \end{pmatrix} \quad \text{for } j \in S_{n-1} \quad \text{and} \tag{18}$$

$$\mathbf{Z}_j = \begin{pmatrix} \omega_j(s) - \left[k_j \partial_x \theta_j^p(x,s) + h_{j+1}\left(\theta_j^p(x,s) - \theta_{j+1}^p(x,s)\right)\right]_{x=x_j} \\ -\left[k_j \partial_x \theta_j^p(x,s) - k_{j+1}\partial_x \theta_{j+1}^p(x,s)\right]_{x=x_j} \end{pmatrix} \quad \text{for } j \in S_{n-1}. \tag{19}$$

The last block element $\mathbf{Z}_n$ on the right hand side and the last diagonal block element $\mathbf{D}_n$ is related to the extreme left and extreme right surfaces L and R

$$\mathbf{Z}_n \equiv \begin{pmatrix} \omega_L(s) - \left[k_L \partial_x \theta_1^p(x,s) - h_L \theta_1^p(x,s)\right]_{x=x_0 \equiv x_L} \\ \omega_R(s) - \left[k_R \partial_x \theta_n^p(x,s) - h_R \theta_n^p(x,s)\right]_{x=x_n \equiv x_R} \end{pmatrix}, \tag{20}$$

$$\mathbf{D}_n \equiv \begin{pmatrix} D_{11}^L & D_{12}^L \\ 0 & 0 \end{pmatrix} \boldsymbol{\mu}_{n-1} \mathbf{M}_{n-1} + \begin{pmatrix} 0 & 0 \\ D_{21}^R & D_{22}^R \end{pmatrix} \quad \text{wher} \tag{21}$$

$$D_{11}^L \equiv \left[-k_L \partial_x \theta_1^-(x,s) + h_L \theta_1^-(x,s)\right]_{x=x_0 \equiv x_L}, \quad D_{12}^L \equiv \left[-k_L \partial_x \theta_1^+(x,s) + h_L \theta_1^+(x,s)\right]_{x=x_L} \tag{22}$$

$$D_{21}^{R} \equiv \left[k_R \partial_x \theta_n^{-}(x,s) + h_R \theta_n^{-}(x,s)\right]_{x=x_n \equiv x_R}, D_{22}^{R} \equiv \left[k_R \partial_x \theta_n^{+}(x,s) + h_R \theta_n^{+}(x,s)\right]_{x=x_R} \tag{23}$$

further

$$\boldsymbol{\mu}_1 = \mathbf{D}_1^{-1}, \quad \boldsymbol{\mu}_j = \left(\mathbf{D}_1^{-1}\mathbf{M}_1\right)\left(\mathbf{D}_2^{-1}\mathbf{M}_2\right)\ldots\left(\mathbf{D}_{j-1}^{-1}\mathbf{M}_{j-1}\right)\mathbf{D}_j^{-1} \quad \texttt{for } 2 \le j \le n. \tag{24}$$

The regularity of all matrices $\mathbf{D}_j$ and $\mathbf{M}_{j\text{-}1}$ follows from the independent ness of θ_j^{-} and θ_{j}^{+} – see (17), (18).

We obtain the solution for $\mathbf{A}_n$ from (15)

$$\mathbf{A}_n = \mathbf{D}_n^{-1}\mathbf{Z}_n. \tag{25}$$

Further we obtain

$$\mathbf{A}_j = \mathbf{D}_j^{-1}\mathbf{Z}_j + \mathbf{D}_j^{-1}\mathbf{M}_{j-1}\mathbf{A}_{j+1} \tag{26}$$

The relation between $\mathbf{A}_1$ and $\mathbf{A}_j$ is given by

$$\mathbf{A}_1 = \boldsymbol{\zeta}_j + \boldsymbol{\mu}_j \mathbf{M}_j \mathbf{A}_{j+1}, \quad j \in S_{n-1} \quad \texttt{where} \tag{27}$$

$$\boldsymbol{\zeta}_j = \boldsymbol{\mu}_1\mathbf{Z}_1 + \boldsymbol{\mu}_2\mathbf{Z}_2 + \ldots + \boldsymbol{\mu}_j\mathbf{Z}_j = \sum_{l \in S_j} \boldsymbol{\mu}_l \mathbf{Z}_l \quad j \in S_n \tag{28}$$

3 Special matrix-form solution with two heat-sources

Consider a multi-layer system of slabs defined in this article with an additional assumption, that the only heat source is on the extreme left or extreme right slab indexed as *L* and *R* respectively. The temperature is equal to zero everywhere in the multi-layer system at $t = 0$. In this case the following conditions are fulfilled

$$\begin{aligned} &F_j(x) \equiv 0, \quad j \in S_n, \\ &g_j(x,t) \equiv 0 \quad \texttt{forall } t > 0 \quad \texttt{and} \quad j \in S_n, \\ &W_j(t) \equiv 0 \quad \texttt{forall } t > 0 \quad \texttt{and} \quad j \in S_{n-1}. \end{aligned} \tag{29}$$

In this case

$$\boldsymbol{\zeta}_j \equiv 0 \quad \texttt{forall } j \in S_{n-1}, \tag{30}$$

therefore:

$$\begin{aligned} &\mathbf{A}_n = \mathbf{D}_n^{-1}\mathbf{Z}_n, \\ &\mathbf{A}_1 = \boldsymbol{\mu}_{n-1}\mathbf{M}_{n-1}\mathbf{A}_n = \boldsymbol{\mu}_{n-1}\mathbf{M}_{n-1}\mathbf{D}_n^{-1}\mathbf{Z}_n \quad \texttt{andfinal} \\ &\mathbf{A}_j = \mathbf{M}_{j-1}^{-1}\boldsymbol{\mu}_{j-1}^{-1}\boldsymbol{\mu}_{n-1}\mathbf{M}_{n-1}\mathbf{D}_n^{-1}\mathbf{Z}_n \quad j = 2,\ldots,n. \end{aligned} \tag{31}$$

4 Conclusion

We present the *matrix-form method* of solving heat conduction in a multi-layered system that consists of n plan-parallel slabs. The great advantage of the matrix-form solutions is evident for multi-layer systems with analogous boundary conditions and initial conditions for all the slabs of the system.

The solution is completed if we know all the $2n$ coefficients A_j^- and A_j^+. The starting point is given by the solution of the equation (25). We obtain all the A-coefficients using the relations (26 – 28). The solution is found in Laplace-space (see 13).

The structure of the solutions reflects the structure of physical properties and conditions in the slabs. The matrices $\mathbf{D}_j$, $\mathbf{M}_j$ and $\mathbf{Z}_j$ represent the physical reality of the connection between adjoining slabs. The matrix-form representation is convenient for modeling by the method of multi-pole approach of the solutions (see [3]), or any other methods.

Acknowledgement

Authors wish to thank the Slovak Science Grant Agency for the financial support under the contract 1/6115/99.

References

[1] Hohenauer W, Vozár L, 1999 An Estimation of Thermophysical Properties of Layered Materials using the Laser Flash Method, *High Temp. High Press*, preprint

[2] Araki N, Makino A, Ishiguro A and Mihara J; 1992 An Analytical Solution of the Temperature Response in Multilayered Materials for Transient Methods; *Int. J. Thermophys.*, **13**, 515-538

[3] Carslaw H S., Jaeger J C, 1959 *Conduction of Heat in Solids*, (Oxford, Clarendon Press)