LINEAR THERMAL EXPANSION OF THE TWO-PHASE CERAMICS

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Abstract: The structure of the two-phase ceramics can be considered as two concentric spheres. The inner sphere, the grain and the outer sphere, the cladding, have radii R_g , R_c and there material parameters are E_g , E_c (Young's moduli), μ_g , μ_c (Poisson's ratios) and α_g , α_c (coefficients of the linear thermal expansion). The expansion of such model during its heating can be solved as a thermoelasticity problem. If the material parameters are constant in the considered temperature region and if $E_g \approx E_c$, $\mu_g \approx \mu_c$ and $\alpha_g \neq \alpha_c$, then the final coefficient of linear thermal expansion of the two phase ceramics depends only on the volume ratio of the grain and the cladding.

Key words: thermal expansion, two-phase solid

1 Introduction

Some ceramic materials can be considered two-phase solids. For example, sintered quartz electrical porcelain contains quartz grains in glassy matrix [1]. We can visualize this structure as consisting of two concentric spheres. The inner sphere, the grain, has a radius R_g , and its material parameters are E_g (Young's modulus), μ_g (Poisson's ratio) and α_g (coefficient of the linear thermal expansion). The outer sphere, the cladding, has material constants E_c , μ_c and α_c and its radius is R_c. We can solve the expansion of this model during its heating as a thermoelasticity problem. If the temperature increase is small and inertial forces are negligible, then the temperature field and the stress field do not influence each other. We assume that grain and cladding materials are homogeneous and isotropic, which means that a radial thermal flow occurs. We also assume that the material parameters are constant in the considered temperature region.

The formula for the coefficient of linear thermal expansion, given the above assumptions, is derived in this contribution.

2 Mathematical model

It follows from the assumptions described above, that the displacement vector **u** has only a radial component u_r. Deformations in a simple sphere along the spherical coordinates are [2]

$$\varepsilon_r = \frac{\partial u_r}{\partial r} \ , \quad \varepsilon_\vartheta = \frac{1}{r} u_r \ , \quad \varepsilon_\varphi = 0$$

and shear deformation is

$$\varepsilon_{r\vartheta} = \frac{1}{2r}\frac{\partial u_r}{\partial \vartheta} = 0 \ ,$$

because the radial displacement u_r does not change along the tangential direction. One can see that deformations are only in the radial and tangential directions. We can use these results for a simple sphere [2]. The radial stress in the grain is

$$\sigma_{rg}(r)=\frac{E_g c_{g1}}{1-2\mu_g}-\frac{2E_g c_{g2}}{1+\mu_g}\frac{1}{r^3}-\frac{2E_g}{1-\mu_g}\frac{1}{r^3}\int_0^r \varepsilon_{gf} r^2 dr \ , \quad \text{for} \ \ 0<r\le R_g \ , \tag{1}$$

and tangential stress is

$$\sigma_{\vartheta g}(r)=\frac{E_g c_{g1}}{1-2\mu_g}-\frac{E_g c_{g2}}{1+\mu_g}\frac{1}{r^3}-\frac{E_g \varepsilon_{gf}}{1-\mu_g}+\frac{E_g}{1-\mu_g}\frac{1}{r^3}\int_0^r \varepsilon_{gf} r^2 dr \ , \quad \text{for} \ 0<r\le R_g \ , \tag{2}$$

and radial component of the displacement vector is

$$u_{rg}(r)=c_{g1}r+c_{g2}\frac{1}{r^2}+\frac{1+\mu_g}{1-\mu_g}\frac{1}{r^2}\int_0^r \varepsilon_{gf} r^2 dr \ , \quad \text{for} \ 0<r\le R_g \ . \tag{3}$$

The value ε_{gf} is a free deformation during the temperature change. This deformation can be calculated using the equation

$$\varepsilon_{gf} = \int_{t_0}^{t} \alpha_g \ dt \ ,$$

thus integral in Eqs. (1), (2), (3) can be written as

$$\int_0^r \int_{t_0}^{t} \alpha_g \ r^2 dr \ dt = \alpha_g \Delta t \frac{r^3}{3} \ ,$$

where $\Delta t = t - t_0$ is a temperature difference between initial temperature t_0 and actual temperature t.

For the cladding we have

$$\sigma_{rc}(r)=\frac{E_c c_{c1}}{1-2\mu_c}-\frac{2E_c c_{c2}}{1+\mu_c}\frac{1}{r^3}-\frac{2E_c}{1-\mu_c}\frac{1}{r^3}\int_{R_g}^r \varepsilon_{cf} r^2 dr \ , \text{for} \ R_g<r\le R_c \ , \tag{4}$$

$$\sigma_{\vartheta c}(r)=\frac{E_c c_{c1}}{1-2\mu_c}-\frac{E_c c_{c2}}{1+\mu_c}\frac{1}{r^3}-\frac{E_c \varepsilon_{cf}}{1-\mu_c}+\frac{E_c}{1-\mu_c}\frac{1}{r^3}\int_{R_g}^r \varepsilon_{cf} r^2 dr \ , \ \text{for} \ R_g<r\le R_c \ , \tag{5}$$

$$u_{rc}(r)=c_{c1}r+c_{c2}\frac{1}{r^2}+\frac{1+\mu_c}{1-\mu_c}\frac{1}{r^2}\int_{R_c}^r \varepsilon_{cf} r^2 dr \ , \ \text{for} \ R_g<r\le R_c \ , \tag{6}$$

and the integral in Eqs. (4), (5), (6) is

$$\int_{R_g}^r \int_{t_0}^{t} \alpha_c \ r^2 dr \ dt = \alpha_c \Delta t \frac{r^3 - R_g^3}{3} \ .$$

Constants c_{g1}, c_{g2}, c_{c1}, c_{c2} can be calculated from the boundary conditions. The boundary conditions on the surface between the grain and cladding ($r = R_g$) and on the surface of the cladding ($r = R_c$) are [3]

$$u_{rg}(0)=0 \ , \quad u_{rg}(R_g)=u_{rc}(R_g) \ , \quad \sigma_{rg}(R_g)=\sigma_{rc}(R_g) \ , \quad \sigma_{rc}(R_c)=0 \ . \tag{7}$$

Substituting Eqs. (1), (3), (4) and (6) into the boundary conditions (7) we obtain four equations

$$\lim_{r\to 0}\left[c_{g1}r+c_{g2}\frac{1}{r^2}+\frac{D_g}{B_g}\frac{1}{r^2}\frac{\alpha_g \Delta t}{3}r^3\right]=0 \ , \tag{8}$$

$$c_{g1}R_g+\frac{D_g R_g}{B_g}\frac{\alpha_g \Delta t}{3}=c_{c1}R_g+\frac{c_{c2}}{R_g^2} \ , \tag{9}$$

$$A_g c_{g1}-2D_g\frac{\alpha_g \Delta t}{3}=A_c c_{c1}-\frac{2B_c}{R_g^3}c_{c2} \ , \tag{10}$$

$$A_c c_{c1} - \frac{2B_c}{R_c^3} c_{c2} - \frac{R_c^3 - R_g^3}{R_c^3} \frac{\alpha_c \Delta t}{3} 2D_c = 0 ,$$

where we introduced abbreviated designations

$$A_g = \frac{E_g}{1-2\mu_g} , \quad A_c = \frac{E_c}{1-2\mu_c} , \quad B_g = \frac{E_g}{1+\mu_g} , \quad B_c = \frac{E_c}{1+\mu_c} , \quad D_g = \frac{E_g}{1-\mu_g} , \quad D_c = \frac{E_c}{1-\mu_c} .$$

By solving these equations we obtain the constants c_{g1}, c_{g2}, c_{c1}, c_{c2} :

$$c_{g1} = \frac{2B_c}{A_g R_g^3}(v-1)c_{c2} + (1-v)\frac{2D_c}{A_g}\frac{\alpha_c \Delta t}{3} + \frac{2D_g}{A_g}\frac{\alpha_g \Delta t}{3} , \tag{11}$$

$$c_{g2} = 0 , \tag{12}$$

$$c_{c1} = \frac{2B_c}{A_g R_g^3} c_{c2} + (1-v)\frac{2D_c}{A_c}\frac{\alpha_c \Delta t}{3} , \tag{13}$$

$$c_{c2} = \frac{2D_c(1-v)(A_g - A_c)\alpha_c - \dfrac{A_c D_g}{B_g}(A_g + 2B_g)\alpha_g}{2A_c B_c (v-1) - 2A_g B_c v - A_g A_c}\frac{\Delta t}{3} R_g^3 , \tag{14}$$

where we introduced the quantity $v = R_g^3 / R_c^3$ representing part of grain in the whole model volume.

Let us take a model with $E_g \approx E_c$, $\mu_g \approx \mu_c$ and $\alpha_g \neq \alpha_c$. Then $A_g = A_c = A$, $B_g = B_c = B$ and $D_g = D_c = D$. Eq. (14) becomes

$$c_{c2} = \frac{D}{3B}\alpha_g \Delta t R_g^3 \tag{15}$$

and from Eqs. (11) and (13) we have

$$c_{c1} = c_{g1} = \frac{2D}{3A}\Delta t\left[v\alpha_g + (1-v)\alpha_c\right] . \tag{16}$$

The radial displacement u_r in the grain (following from Eqs. (3), (15) and (16)) is

$$u_{rg}(r) = \frac{D\Delta t}{3A} r\left[\frac{A+2Bv}{B}\alpha_g - 2(1-v)\alpha_c\right] \tag{17}$$

and for the radial and tangential stress we have from Eqs. (1), (2), (15) and (16)

$$\sigma_{rg} = \sigma_{\vartheta g} = -\frac{2D\Delta t}{3}(\alpha_g + \alpha_c)(1-v) , \tag{18}$$

which means that the grain was exposed to hydrostatic pressure. The cladding does not allow free expansion of the grain.

The radial displacement in the cladding is

$$u_{rc}(r) = \frac{D\Delta t}{3}\left[\left(\frac{2rv}{A} + \frac{R_g^3}{Br^2}\right)\alpha_g + \left(\frac{2r}{A} + \frac{r}{B} - \frac{2rv}{A} - \frac{R_g^3}{Br^2}\right)\alpha_c\right] \tag{19}$$

as it follows from Eqs. (6), (15) and (16). The radial and tangential stress is

$$\sigma_{rc}(r) = \frac{2D\Delta t}{3}\left(v - \frac{R_g^3}{r^3}\right)(\alpha_g - \alpha_c) , \tag{20}$$

$$\sigma_{\vartheta c}(r) = \frac{D\Delta t}{3}\left(2v + \frac{R_g^3}{r^3}\right)(\alpha_g - \alpha_c) . \tag{21}$$

Eqs. (20) and (21) show that the stress on the boundary between the grain and the cladding does not depend on the grain radius.

Let us change now the model described above with a homogeneous sphere made of fictive material with coefficient of the linear thermal expansion α_f. The size of this sphere at the temperature t_0 is the same as the composite sphere, i.e. its radius is R_c. Both spheres are equivalent if their radial displacements are equal at the temperature t

$$\{u_{rg}(R_g) + u_{rc}(R_c)\}_{comp.sphere} = \{u_r(R_c)\}_{fict.sphere} \ ,$$

where $\{u_r(R_c)\}_{fict.sphere}$ can be calculated from the equation similar to the Eq. (3) and boundary conditions $u_{rf}(0) = 0$, $\sigma_{rf}(R_c) = 0$. Then we obtain

$$D\left[\left(\frac{2v}{A} + \frac{R_g^3}{BR_c^3}\right)\alpha_g + \left(\frac{2}{A}(1-v) + \frac{1}{B}\left(1 - \frac{R_g^3}{R_c^3}\right)\right)\alpha_c\right]R_c \frac{\Delta t}{3} = 2D\left(\frac{1}{A} + \frac{1}{2B}\right)R_c\alpha_f \frac{\Delta t}{3}$$

and after mathematical modifications

$$\alpha_f = v\alpha_g + (1-v)\alpha_c \ . \qquad (22)$$

In the simple case when material parameters of the grain and the cladding are the same except for the coefficients of linear thermal expansion, the final coefficient of linear thermal expansion depends only on the volume ratio of the grain and the cladding.

We note that Eq. (22) satisfies the extreme cases: if v = 1 (model contents only grain), we have $\alpha_f = \alpha_g$ and if v = 0 (model contents only cladding), we have $\alpha_f = \alpha_c$.

3 Conclusions

The structure of the two-phase ceramics can be considered as two concentric spheres model. The expansion of such model during its heating can be solved as a thermoelasticity problem. If the material parameters of the inner sphere and outer sphere are constant in the considered temperature region and if there elastic moduli and Poisson's ratios are very close to each other and the coefficients of the linear thermal expansion are different, then the final coefficient of the linear thermal expansion depends only on the volume ratio of the inner and outer spheres.

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