THICKNESS EFFECT CURVE IN THE ASPECT OF ANALYTICAL AND NUMERICAL EVALUATIONS

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Abstract

This paper concerning the problem of analytical and numerical modelling of ‘thickness effect curve’ under steady-state conditions for interactive optical media, where simultaneous radiation and conduction heat transfer occurs. Considerations are performed for a 1-D steady-state heat transfer model in an absorbing, emitting and anisotropically scattering grey medium confined by grey surfaces. To find dependence of the radiative thermal conductivity $k_r(l)$ on the sample thickness l, a finite difference method (FDM) together with a discrete ordinate method (DOM) and the Henyey-Greenstein phase function were used iteratively.

Key words: radiative-conductive heat transfer, thermal conductivity, absorption, emission and scattering of radiation, thickness effect curve, phase function

1 Introduction

The results of modelling and numerical simulation of the effect of reduction in thermal conductivity $k(l)$ for small thickness samples of semitransparent media in case when enclosing the medium surfaces are grey and diffuse are reported. The subject of an interest is isotropic and homogeneous material with thermal conductivity dependence $k(l)$ on the sample thickness l obtained as a result of measuring it under conditions of a definite value (a few degrees) of the temperature difference ΔT between the heater T_2 and the cooler T_1 of the plate apparatus – Figs. 1÷2. The visible in Fig. 2 two points marked as A and B correspond to the two different sample thicknesses l_A and l_B.

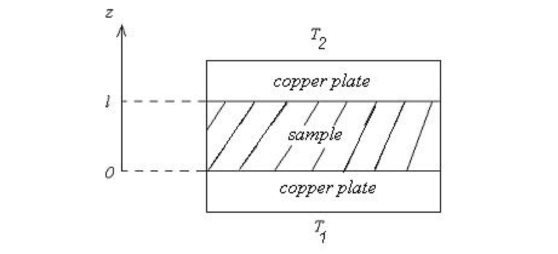

Fig. 1 The sample of thickness l between the heater T_2 and the cooler T_1 of the plate apparatus

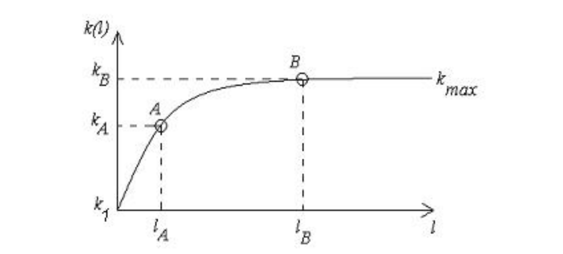

Fig. 2 Effect of reduction in thermal conductivity $k(l)$ in small thickness sample: k_1 - conductive component of thermal conductivity, $k_r(l) = k(l) - k_1$

2 Analytical model

Let us assume that the material with different forms of the curve of thermal conductivity $k(l)$ dependence on the sample thickness l having also different values of the limiting sample thickness l_{gr}, above which the reduction of $k(l)$ may be neglected, is considered. The jump-like relation $k(z)$, presented in Figs 3 and 4, shows the model dependence on the thermal conductivity $k(z)$ (for a given sample thickness l) [1] from which it follows that $k = k_1 + k_r$ where, in particular case, at z=0 and T=T_1 we have $k(z=0) = k_1$. Similarly, for the other side of the sample at z=l and T=T_2 we have also $k(z=l) = k_1$ (Figs. 3 and 4). Each curve $k(z)$ can be divided into three parts, the first and the third of which correspond to the region from z=0 to z=l_{gr} where $k = k_1$ and from z=l-l_{gr} to z=l where $k = k_1$, too. The second corresponds to the region from z=l_{gr} to z=l-l_{gr} where $k = k_2$. Now it is possible to determine $k(l)$ dependence for a sample divided into three layers, where the thickness of the particular layer equals l_{gr}, l-$2l_{gr}$, l_{gr}, respectively, by solving the linear 1-D steady-state heat conduction problem for each of the particular region [1]:

$$k(l) = \frac{k_1 k_2 l}{2 l_{gr} k_2 + (l - 2l_{gr}) k_1} = \frac{k_1 k_2}{2\frac{l_{gr}}{l}(k_2 - k_1) + k_1}, \tag{1}$$

where l_{gr} is a function of absorption coefficient a and sample thickness l. Then the radiative heat flux density $\dot{q}_r(l)$ can be calculated from the following expression [1]:

$$\dot{q}_r(l) = k_r \frac{dT_{II}}{dz_1}\frac{(l - 2l_{gr})}{l} = (k_2 - k_1)\cdot\frac{T_2 - T_1}{(l - 2l_{gr})\left(1 - \frac{k_2}{k_1}\right) + \frac{k_2}{k_1} l}\cdot\frac{(l - 2l_{gr})}{l} \tag{2}$$

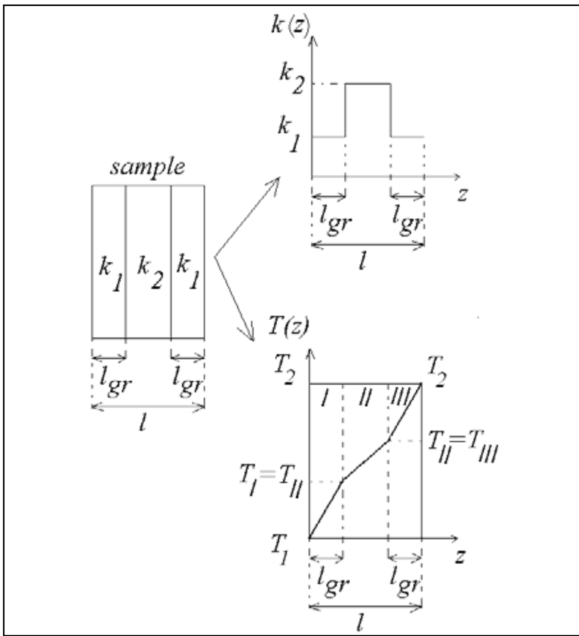

Fig. 3. The model of dependence of $k(z)$ (for the sample thickness l) accepted for analytical considerations [1]

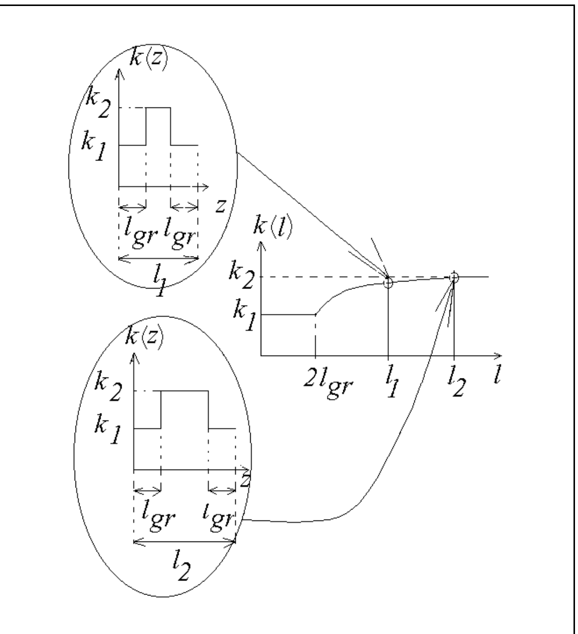

Fig. 4. Plot of $k(l)$ along different sample thickness l with marked two points corresponding to the two different sample thicknesses l_1 and l_2 [1]

When the sample thickness l is getting closer to $l=2l_{gr}$, the mean free path of photons becomes comparable to the space between plates and some of the photons leaving the hot surface manage to reach almost or actually the cold surface before they become absorbed. In our opinion such a situation permits us to replace l_{gr} with expression $l'_{gr} = l_{gr}\left(1 - e^{-a\cdot b\cdot l}\right)$, where b is a constant parameter [1]. After doing some algebraic manipulations, one can obtain the following condition for the limiting thickness of the sample l_{gr} as well as for a new form of expressions for $k(l)$ and $\dot{q}_r(l)$ [1]:

$$l_{gr} = \frac{k_1}{3ak_1 + 4\sigma_B n^2 (T_1 + T_2)(T_1^2 + T_2^2)}, \tag{3}$$

$$k(l) = \frac{k_1(k_1 + k_{r\infty})}{2\frac{l_{gr}}{l}\left(1 - e^{-\frac{l}{2l_{gr}}}\right)k_{r\infty} + k_1} \tag{4}$$

$$\dot{q}_r(l) = \frac{(T_2 - T_1)k(l)\left[k_{r\infty} - k_1\left(\frac{k_1 + k_{r\infty}}{k(l)} - 1\right)\right]}{l(k_1 + k_{r\infty})}, \tag{5}$$

$$(k(l))_{l\to\infty} = k_1 + k_{r\infty} = k_2\,, \tag{6}$$

$$k_{r\infty} = \frac{16\sigma_B n^2}{3a}\cdot\frac{1}{4}(T_1 + T_2)(T_1^2 + T_2^2), \tag{7}$$

where: σ_B is the Stefan-Boltzmann constant ($\sigma_B = 5.67\cdot 10^{-8}\ \mathrm{W\,m^{-2}\,K^{-4}}$), n is the index of refraction, k_1 denotes the conductive component of thermal conductivity [5-8]. Some preliminary calculations of $k(l)$ and $\dot{q}_r(l)$ obtained both from the numerical solution of the Radiative Transfer Equation (RTE – exact formulation) coupled with 1-D heat conduction in an emitting-absorbing medium and from analytical model are shown in Figs. 5 and 6 [1].

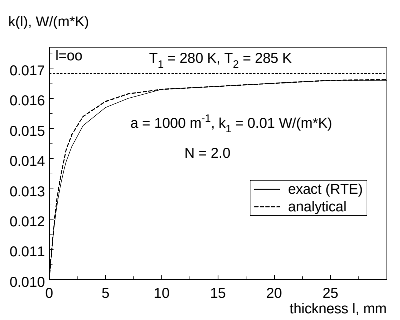

Fig. 5. Thermal conductivity *k(l)* obtained from RTE vs. analytical model [1]

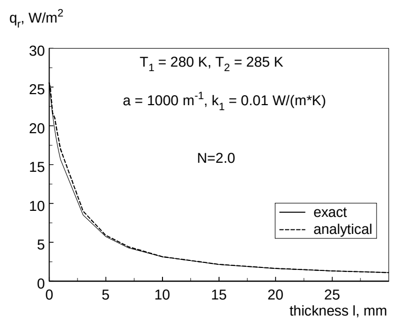

Fig. 6. Radiative heat flux density $\dot{q}_r(l)$ obtained from RTE vs. analytical model [1]

The biggest discrepancy between the precise numerical results and the given model occurs when the conduction to radiation parameter N equals 0.02 [1]. It is not surprising that in that case the biggest non-linearity of temperature distribution $T(z)$ inside the sample occurs along its thickness. It has to be underlined that all calculations and analytical model have been made for emissivity of the walls $\varepsilon_1=\varepsilon_2=\varepsilon=1$. When the surfaces enclosing the medium are grey and are diffuse reflectors, the results of calculations which were made to include the influence of the surface emissivity on the $k(l)$ are illustrated in Fig. 7. In case of very low emissivity of the wall one can observe that curves $k(l)$ reveal characteristic inflexion in the range of small thickness of the samples l corresponding to the maximum of curves $\dot{q}_r(l)$. The analytical model takes into account both inflexion of curves $k(l)$ as well as maximum of $\dot{q}_r(l)$. In the vicinity of the sample surface the energy transferred by conduction is a larger fraction of the total energy flux than in the region located farther from the walls. Thus temperature gradient in the neighborhood of the walls increase with the decreasing emissivity, and therefore the conductive flux in these regions becomes a larger fraction of the total energy flux. In our opinion such situation permits us to replace l by $l' = l + 2l'_{gr}$ and l'_{gr} by the expression $l'_{gr} = l''_{gr}\left(1-e^{-a\cdot l'}\right)$ [2]. Just like before performing some algebraic rearrangements one can obtain the following condition for the limiting thickness of the sample l''_{gr}

$$l''_{gr} = \frac{1-\sqrt{\dfrac{l_{gr}\left[3a+\dfrac{4\sigma_B n^2}{k_1}\left(T_1+T_2\right)\left(T_1^2+T_2^2\right)\right]}{\dfrac{2}{\varepsilon}-1}}}{2a} = \frac{1-\sqrt{\dfrac{\varepsilon}{2-\varepsilon}}}{2a} \tag{8}$$

and a new form of expressions for $k(l)$ and $\dot{q}_r(l)$ [2]:

$$k(l) = \frac{k_1\left(k_1+k_{r\infty}\right)lG(l)}{2l''_{gr}\left(1-e^{-a\cdot l}\right)\left\{\left(k_1+k_{r\infty}\right)G(l)-H(l)\right\}+lH(l)} \tag{9}$$

$$G(l) = l - 2l''_{gr}\left(1-e^{-a\cdot l}\right),\ H(l) = 2l_{gr}\left(1-e^{-\frac{G(l)}{2l_{gr}}}\right)k_{r\infty} + k_1 G(l),\ \dot{q}_r(l) = \left(k(l)-k_1\right)\frac{\Delta T}{l} \tag{10}$$

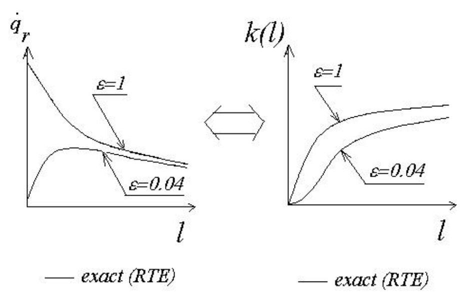

Fig. 7. Plot of thermal conductivity $k(l)$ and radiative heat flux density $\dot{q}_r(l)$ along different sample thickness l according to RTE calculations with surface emissivity less than one [2]

In order to verify the proposed method some preliminary calculations of $k(l)$ and $\dot{q}_r(l)$ were carried out [2]. Dependencies of $k(l)$ and $\dot{q}_r(l)$ on the sample thickness l obtained from RTE and from analytical model are shown in Figs. 8 and 9. With the increase of the parameter N the discrepancies between the analytical functions of $k(l)$, $\dot{q}_r(l)$ and the exact results decrease and for N=2 the errors are acceptable [2].

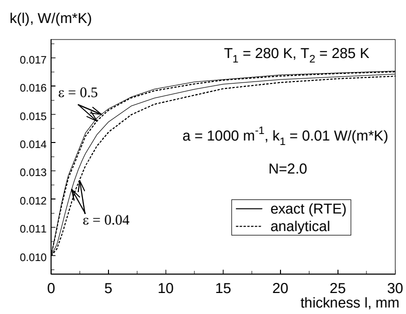

Fig. 8. Dependence of *k(l)* obtained from RTE and analytical model: $\varepsilon_1=\varepsilon_2=\varepsilon$ = 0.5 or 0.04 [2]

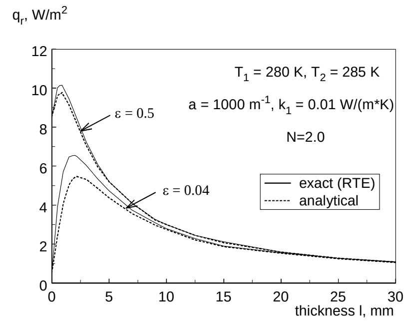

Fig. 9. Dependence of $\dot{q}_r(l)$ obtained from RTE and analytical model: $\varepsilon_1=\varepsilon_2=\varepsilon$ = 0.5 or 0.04 [2]

3 Numerical model

A model of 1-D steady-state combined conductive-radiative heat transfer in the absorbing, emitting and anisotropically scattering medium confined between grey surfaces has been considered. To find the radiative thermal conductivity dependence $k_r(l)$ (for a sample thickness l) and dependence of the radiative heat flux density $\dot{q}_r(l)$ upon the sample thickness l, a finite difference method together with the discrete ordinate method and the Henyey-Greenstein phase function expanded into Legendre polynomials have been used iteratively [3, 4]. The Henyey-Greenstein phase function $P(\cos\Theta)$, where Θ denotes the scattering angle (angle between incident and scattered direction of radiation), allows us to understand the influence of the asymmetry parameter g on the shape of the scattering phase function [4]. When the asymmetry parameter g increases, the anisotropic scattering radiation increases too and, at the same time, the range of influence of scattering extends. Parameter of asymmetry g defines forward scattering for g=1, backward scattering for g=-1 and isotropic scattering for g=0. Other values of the asymmetry parameter g which belong to the interval $g\in[-1; 1]$ are also possible [3, 4]. In our considerations it is assumed that the conductive component of thermal conductivity $k_1 = k_c$ is constant. The radiative properties of the medium such as the absorption coefficient a, extinction coefficient κ and index of refraction n are also constant. The governing equation for a non-grey medium which represents energy conservation in 1-D formulation of the coupled conduction-radiation problem is given by [3, 4, 5-8]:

$$\frac{d}{dz}\left(k_c \frac{dT}{dz}\right) = \frac{d\dot{q}_r(z)}{dz},\ 0 < z < l \tag{11}$$

$$T(z=0) = T_1, \tag{12}$$

$$T(z=l) = T_2, \tag{13}$$

$$\dot{q}_r(z) = 2\pi \int_0^{\infty} \int_{\mu=-1}^{\mu=+1} I_\lambda(z,\mu)\,\mu\,\mathrm{d}\mu\,\mathrm{d}\lambda, \tag{14}$$

$$\mu \frac{dI_\lambda(z,\mu)}{dz} = a_\lambda(\mu)\ I_{b,\lambda}(T(z)) - (\kappa_\lambda(\mu))\,I_\lambda(z,\mu) + \frac{1}{2}\int_{-1}^{+1} \sigma_{s,\lambda}(\mu')P(\mu' \to \mu)I_\lambda(z,\mu')d\mu' \tag{15}$$

$$I_\lambda(0,\mu) = \varepsilon_1 I_{b,\lambda}(T_0) + 2(1-\varepsilon_1)\int_{-1}^{0} I_\lambda(0,\mu')\mu' d\mu', \quad 0 < \mu \le 1 \tag{16}$$

$$I_\lambda(l,\mu) = \varepsilon_2 I_{b,\lambda}(T_l) + 2(1-\varepsilon_2)\int_{0}^{+1} I_\lambda(l,\mu')\mu' d\mu', \quad -1 \le \mu < 0 \tag{17}$$

$$I_{b,\lambda} = \frac{C_1}{\lambda^5\left[\exp(C_2/\lambda T) - 1\right]},\ I_b(T) = \int_{\lambda=0}^{\infty} I_{b,\lambda}(T)d\lambda = \frac{1}{\pi}\sigma_B T^4, \tag{18}$$

$$C_1 = 1.19 \cdot 10^{-16}\ \mathrm{W \cdot m^2},\ C_2 = 0.014388\ \mathrm{m \cdot K}, \tag{19}$$

$$\kappa(\mu) = a(\mu) + \sigma_S(\mu),\ \mu = \cos\theta, \tag{20}$$

where: I_λ is the spectral radiation intensity, Wm^{-2}; $I_b = n^2\sigma_B T^4/\pi$ stands for the blackbody intensity, Wm^{-2}; θ is the polar angle and σ_s is the scattering coefficient, m^{-1} [5-8].

In order to elucidate the physics of the analyzed complex phenomenon of simultaneous conduction and radiation in an emitting, absorbing and anisotropically scattering medium confined between grey surfaces some preliminary calculations of $k_r(l)$ and $\dot{q}_r(l)$ were carried out [3, 4]. The initial data for performing numerical calculations are enclosed in Tab. 1. Data used in calculations were chosen in such a way that the impact of albedo ω and the parameter of asymmetry g on $k_r(l)$ and $\dot{q}_r(l)$ thickness dependences are visible. Finally the calculations were performed using 6 fluxes in the conductive-radiative heat transfer model. Six fluxes turned out, in this case, to be quite enough to obtain an engineering accuracy of calculations [4]. The dimensionless conduction to radiation parameter N=8 was taken big enough so as to avoid instability of numerical calculations. The results of calculation are shown in Figs. 10÷13 [3, 4].

Table 1. Data for performing numerical calculations of $k_r(l)$ and $\dot{q}_r(l)$ [3, 4]

$\kappa = 1000\ m^{-1}$, $k_1 = k_c = 0.1\,Wm^{-1}K^{-1}$, $\omega \in [0;1]$, n=1, T_1=280 K, T_2=380 K, N=8	
$\varepsilon_1 = \varepsilon_2 = \varepsilon = 1$	$\varepsilon_1 = \varepsilon_2 = \varepsilon = 0.04$
$\omega = \in \{0.0,\ 0.3,\ 0.5,\ 0.9,\ 1.0\}$, g=-0.8 or g=+0.8	
$P(\cos\Theta) = (1 - g^2)/(1 + g^2 - 2g\cos\Theta)^{3/2}$	
ε - emissivity of the walls, $\omega = \sigma_S / \kappa$ - single scattering albedo, $\kappa = a + \sigma_S$ - extinction coefficient, m^{-1}, σ_S - scattering coefficient, m^{-1}	

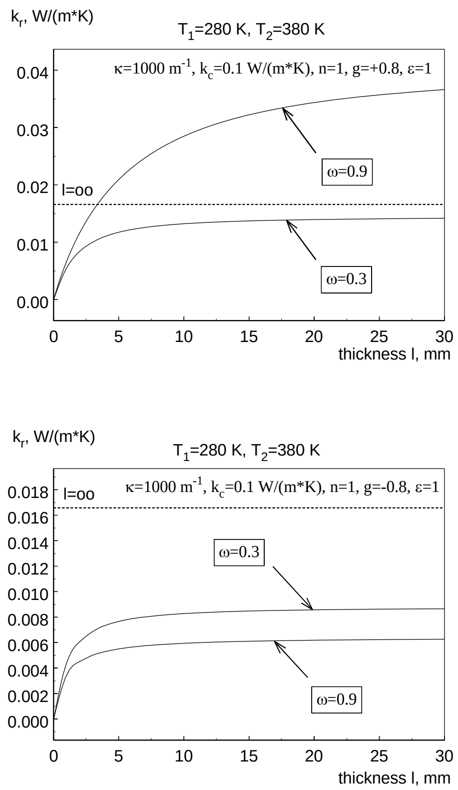

Fig. 10. Dependence of $k_r(l)$ for anisotropic scattering: g=±0.8, $\omega \in \{0.3, 0.9\}$, $\varepsilon_1=\varepsilon_2=\varepsilon$ = 1.0 [4]

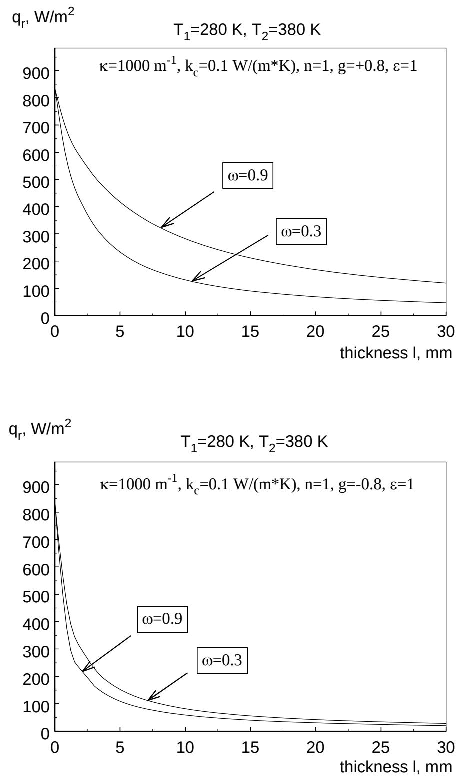

Fig. 11. Dependence of $\dot{q}_r(l)$ for anisotropic scattering: g=±0.8, $\omega \in \{0.3, 0.9\}$, $\varepsilon_1=\varepsilon_2=\varepsilon$ = 1.0 [4]

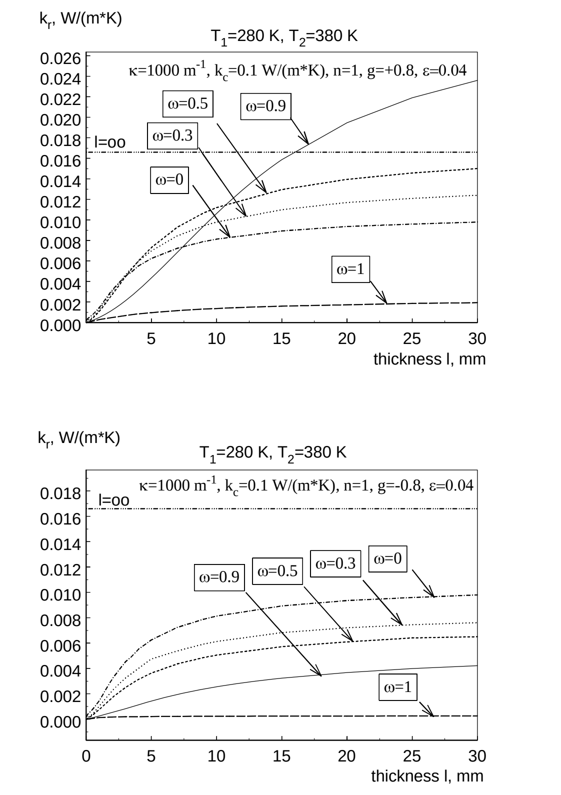

Fig. 12. Dependence of $k_r(l)$ for anisotropic scattering: $g=\pm0.8$, $\omega \in \{ 0.0, 0.3, 0.5, 0.9, 1.0\}$, $\varepsilon_1=\varepsilon_2=\varepsilon = 0.04$ [4]

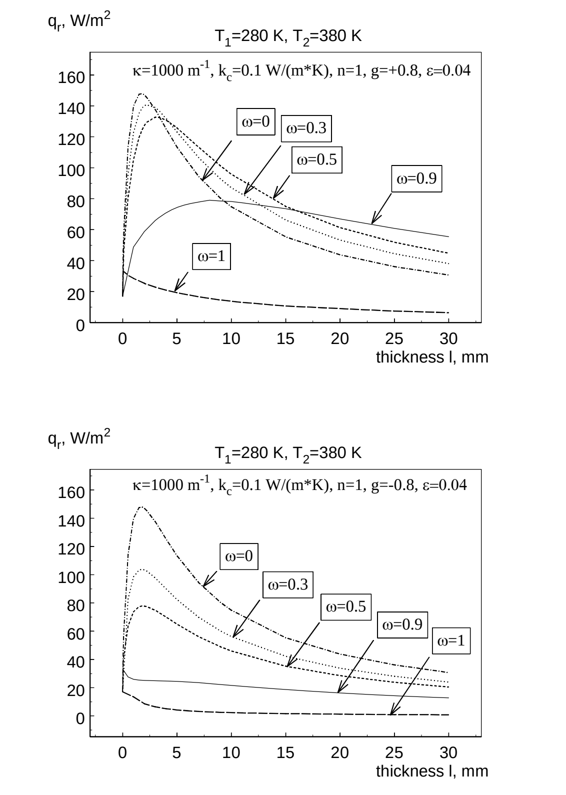

Fig. 13. Dependence of $\dot{q}_r(l)$ for anisotropic scattering: $g=\pm0.8$, $\omega \in \{ 0.0, 0.3, 0.5, 0.9, 1.0\}$, $\varepsilon_1=\varepsilon_2=\varepsilon = 0.04$ [4]

The most interesting situation occurs when anisotropic scattering is considered [3, 4]. The higher value of albedo ω the more important role plays the parameter of asymmetry g. And so for $\omega = 0.3$ the influence of the coefficient g on the $k_r(l)$ and $\dot{q}_r(l)$ dependencies is relatively small – Figs. 10÷13. The curves $k_r(l)$ are places below line l_∞ - Figs. 10 and 12. For $g = +0.8$ that is for strong scattering the curves $k_r(l)$ are close to the values $k_{\max} = k_{Ross} = \dfrac{16 n^2 \sigma_B T^{*3}}{3\kappa}$ (where: $T^*=T_2$) marked as l_∞ - Figs. 10 and 12 [3, 4]. For $g = -0.8$ curves $k_r(l)$ are significantly below the line l_∞ [3, 4]. The situation changes totally when the albedo is equal to $\omega = 0.9$. For $g = +0.8$ the curves $k_r(l)$ lie significantly above the line l_∞ [3, 4]. This case plays a fundamental role in real experiment. It turns out that the values of radiative component of thermal conductivity $k_r(l)$ may be measurable and the role of heat transfer by radiation, even in relatively low temperatures, might be significant.

4 Conclusions

The results of modelling and numerical simulation of the effect of reduction in thermal conductivity $k(l)$ in small thickness samples of semitransparent media have inspired the authors to develop a possible precise enough analytical and numerical model for the abovementioned effect. It is worth noticing that the analytical model works qualitatively well [1÷4]. In case of numerical evaluation the stress was put on the method of solving the coupled radiative-conductive heat transfer problem using FDM to find temperature fields along the layer thickness of considered material together with DOM to find the radiation intensity which were used to calculate the source term appearing in the governing equation of energy conservation. The analysis of the obtained numerical results enabled to draw practical conclusions concerning the experimental measurements with respect to the effect of reduction in thermal conductivity $k(l)$.

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