Models for measurement of thermophysical parameters

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Model as a image of real experiment

Heat equation:

$$c\rho\frac{\partial T}{\partial t} = \nabla \cdot \lambda \nabla T \tag{1}$$

Initial and boundary conditions:

$$adequate\ to\ the\ experimental\ setup \tag{2}$$

Solution:

$$T(t,\mathbf{r}) \sim T_{EXP}(t,\mathbf{r})$$

c ... Specific heat capacity

ρ ... Density

λ ... Thermal conductivity

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Specification of the problem

- The parameters are temperature independent ⇒Linearisation of the problem
- The sample have certain symmetry ⇒Possibility of spatial dimension reduction
- The heat source is external ⇒Time dependent boundary conditions

Used methods

- Laplace transform for time dependence
- Specific integral transform for spatial dependence
- Superposition method

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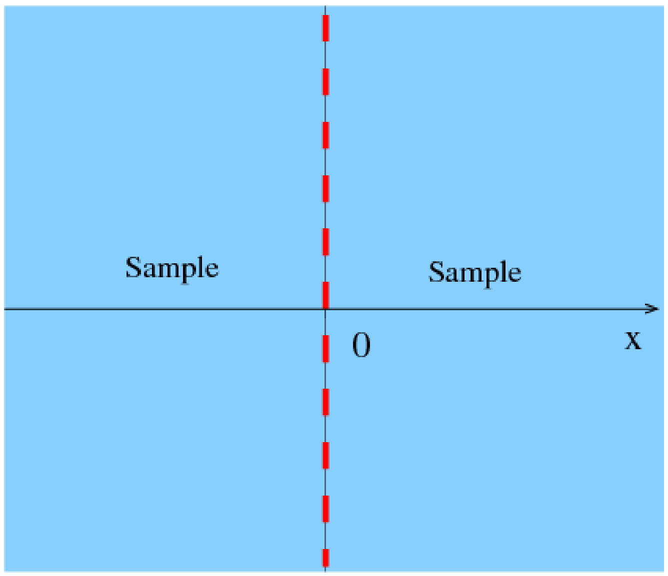
Solved models - 1D



1. Ideal model. Instantaneous planar heat source placed between two semiinfinit samples. The thermal contact of heat source and samples is ideal . Heat equation:

$$\frac{1}{k}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} \tag{3}$$

Initial condition:

$$T(0, x) = 0 \tag{4}$$

Boundary conditions:

$$-\lambda \frac{\partial T}{\partial x}\bigg|_{x=0} = q1(t) \tag{5}$$

$$T(t, \infty) = 0 \tag{6}$$

Solution

$$T(t, x) = T_0 \left[\frac{e^{-u^2}}{\sqrt{\pi} u} - \Phi^*(u) \right]$$

$$u = \frac{x}{2\sqrt{kt}} \qquad T_0 = \frac{qx}{\lambda}$$

T ... temperature
t ... time
x ... Cartesian coordinate
q ... heat flow density at source
$1(t)$ ... Heaviside unit step function
λ ... thermal conductivity
k ... thermal diffusivity
C ... heat capacity per unit area of source
α_s ... heat trasfer coefficient for sample - heat source interface
$\Phi^*(u)$ is the complementary error function .

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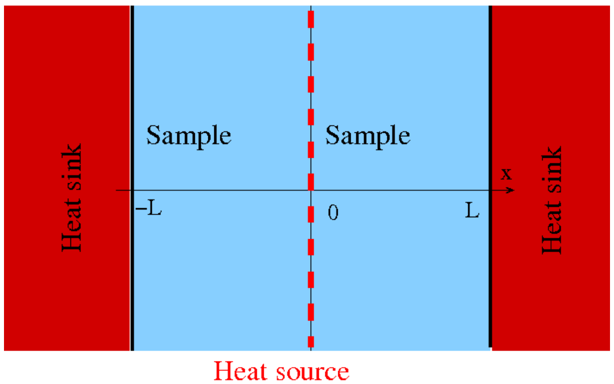
2. Slabs - symmetric disposition



Boundary conditions:

$$-\lambda \frac{\partial T}{\partial x}\bigg|_{x=0} = \alpha_s (T_s - T\big|_{x=0}) \qquad (7)$$

$$q1(t) = C\frac{\partial T_s}{\partial t} + \alpha_s (T_s - T\big|_{x=0}) \qquad (8)$$

$$T(t, L) = 0 \qquad (9)$$

Solution

$$T(t,x) = T_0 \left\{ (1 - \tfrac{x}{L}) + 2a \sum_{\nu} e^{-\frac{kt}{L^2}\nu^2} \times \frac{\nu \sin(\nu \frac{x}{L}) - (a - b\nu^2)\cos(\nu \frac{x}{L})}{\nu^2 [b^2\nu^4 - (2ab - b - 1)\nu^2 + a(a+1)]} \right\}$$

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$$T_0 = \frac{qL}{\lambda} \qquad a = \frac{\lambda L}{Ck} \qquad b = \frac{\lambda}{L\alpha_s}$$

T_s ... temperature of heat source
L ... thickness of sample
C ... heat capacity per unit area of source
α_s ... heat transfer coefficient of source - sample interface
ν is a root of equation

$$(a - b\nu^2)\cos\nu - \nu\sin\nu = 0$$

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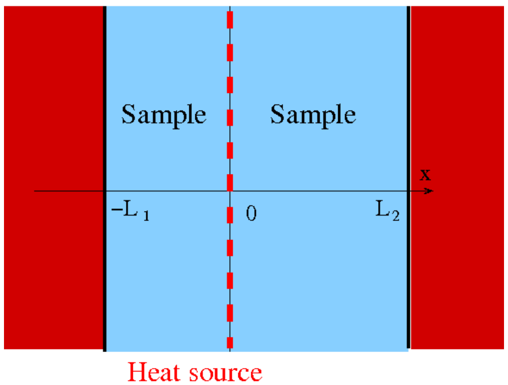
3. Slabs - asymmetric disposition



Boundary conditions:

$$-\lambda \frac{\partial T}{\partial x}\bigg|_{x=0^+} + \lambda \frac{\partial T}{\partial x}\bigg|_{x=0^-} = 2q1(t) \quad (10)$$

$$T|_{x=0^+} - T|_{x=0^-} = 0 \quad (11)$$

$$T|_{x=-L_1} = T|_{x=L_2} = 0 \quad (12)$$

Solution

$$T(t,x) = \frac{2q\sqrt{kt}}{\lambda} \sum_{j=-\infty}^{\infty} (-1)^j \left[\frac{e^{-u_j^2}}{\sqrt{\pi}} - u_j \Phi^*(u_j)\right]$$

$$u_j = \frac{||x| + 2Lj + \frac{1}{2}(L_1 - L_2)[1 - (-1)^j]|}{2\sqrt{kt}}$$

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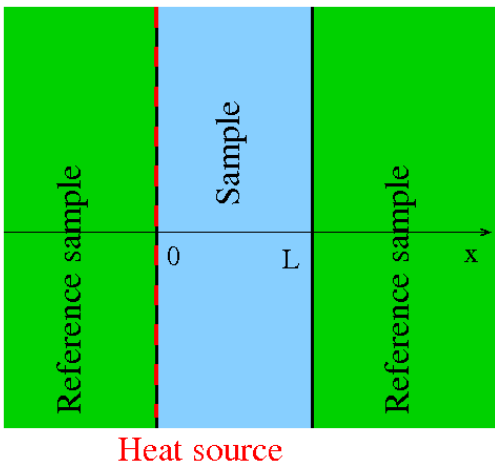
L_1, L_2 ... thicknesses of samples
$L = \frac{1}{2}(L_1 + L_2)$
$2q$ ... sum of heat flow densities at source

4. Sandwich disposition



Boundary conditions:

$$T(t, -\infty) = 0 \tag{13}$$

$$T(t, \infty) = 0 \tag{14}$$

$$-\lambda \frac{\partial T}{\partial x}\bigg|_{x=0^+} + \lambda_0 \frac{\partial T}{\partial x}\bigg|_{x=0^-} = 2q1(t) \tag{15}$$

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$$T|_{x=0^+} - T|_{x=0^-} = 0 \tag{16}$$

$$-\lambda \frac{\partial T}{\partial x}\bigg|_{x=h^-} + \lambda_0 \frac{\partial T}{\partial x}\bigg|_{x=h^+} = 0 \tag{17}$$

$$T|_{x=h^-} - T|_{x=h^+} = 0 \tag{18}$$

Solution

$$T_{13}(t,x) = T_0 \frac{\gamma}{u_0} \sum_{j=0}^{\infty} \delta^j \left[\frac{e^{-u_j^2}}{\sqrt{\pi}} - u_j \Phi^*(u_j) \right]$$

$$T_0 = \frac{qh}{\lambda} \quad u_j = \frac{(n+\frac{1}{2})h}{\sqrt{kt}}, \quad \gamma = \left(\frac{2}{1 + \frac{\lambda_0}{\lambda}\sqrt{\frac{k}{k_0}}} \right)^2, \quad \delta = \left(\frac{1 - \frac{\lambda_0}{\lambda}\sqrt{\frac{k}{k_0}}}{1 + \frac{\lambda_0}{\lambda}\sqrt{\frac{k}{k_0}}} \right)^2$$

h ... thickness of the sample
λ_0 ... reference thermal conductivity
k_0 ... reference thermal diffusivity

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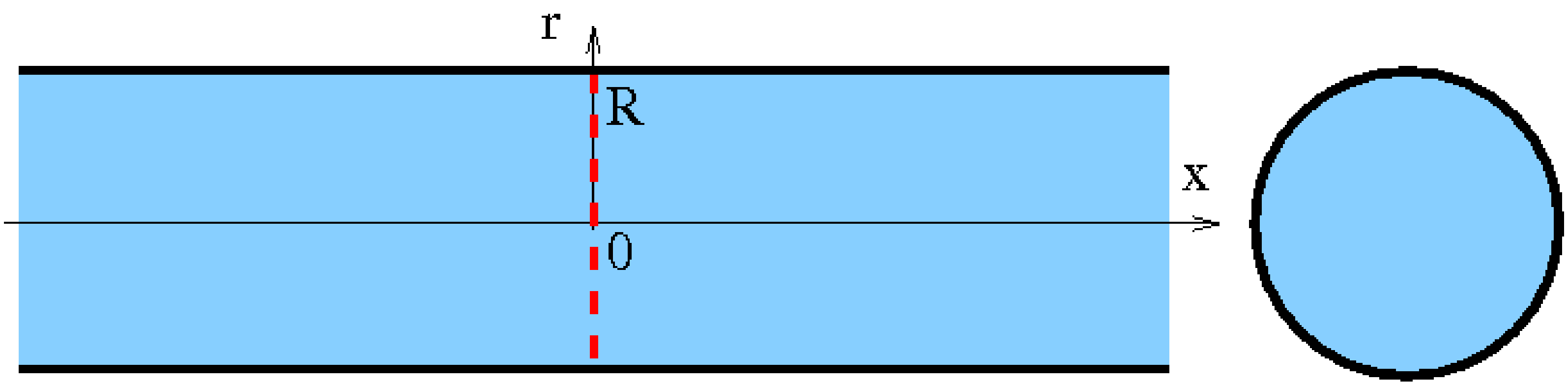
Solved models - 2D

5. Semi-infinite cylinder



Heat equation:

$$\frac{1}{k}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial T}{\partial r} \tag{19}$$

Initial condition:

$$T(0, x) = 0 \tag{20}$$

Boundary conditions:

$$-\lambda \frac{\partial T}{\partial r}\bigg|_{r=R} = \alpha T\bigg|_{r=R} \tag{21}$$

$$-\lambda \frac{\partial T}{\partial x}\bigg|_{x=0} = q1(t) \tag{22}$$

$$T\bigg|_{x=\infty} = 0 \tag{23}$$

Solution

$$T(t,x,r) = T_0 \tfrac{R}{x} \sum_{\xi} \tfrac{\beta}{\xi(\xi^2+\beta^2)} \tfrac{J_0(\xi \frac{r}{R})}{J_0(\xi)} F(u,v)$$

$$F(u,v) = e^{-2uv}\Phi^*(u-v) - e^{2uv}\Phi^*(u+v)$$

$$T_0 = \frac{qx}{\lambda} \qquad \beta = \frac{R\alpha}{\lambda} \qquad u = \frac{x}{2\sqrt{kt}} \qquad v = \xi \frac{\sqrt{kt}}{R}$$

x ... axial space coordinate
r ... radial space coordinate
R ... radius of the sample
q ... heat flow density at source
α ... heat trasfer coefficient for sample - ambient interface
ξ is the root of the equation

$$\beta J_0(\xi) - \xi J_1(\xi) = 0$$

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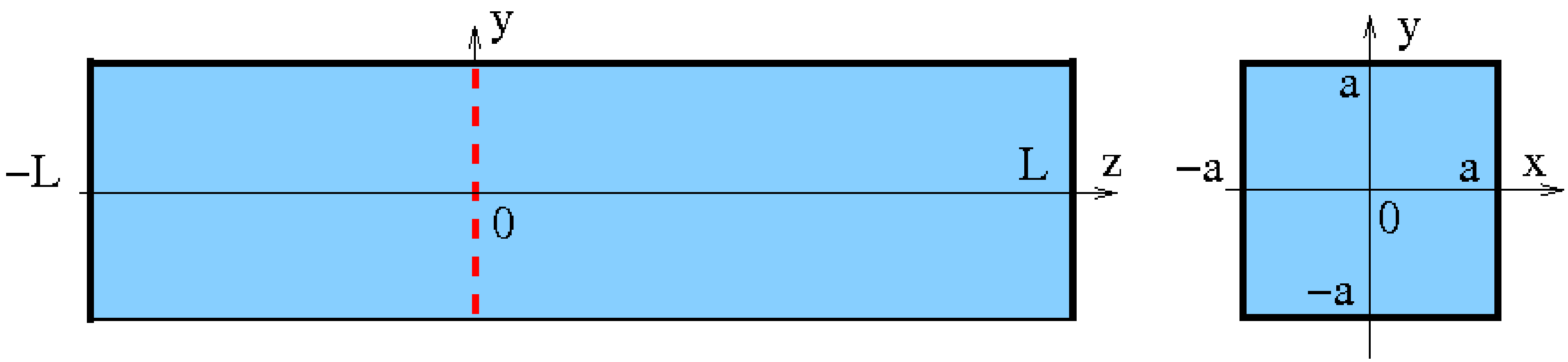
Solved models - 3D

5. Finite cuboid



Heat equation:

$$\frac{1}{k}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \tag{24}$$

Initial condition:

$$T(0, x, y, z) = 0 \tag{25}$$

Boundary conditions:

$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0 \tag{26}$$

$$-\lambda\left.\frac{\partial T}{\partial x}\right|_{x=a} = \alpha T\Big|_{x=a} \tag{27}$$

$$\left.\frac{\partial T}{\partial y}\right|_{y=0} = 0 \tag{28}$$

$$-\lambda\left.\frac{\partial T}{\partial y}\right|_{y=a} = \alpha T\Big|_{y=a} \tag{29}$$

$$T\Big|_{z=-L_1} = T\Big|_{z=L_2} = 0 \tag{30}$$

$$T\Big|_{z=0^+} = T\Big|_{z=0^+} \tag{31}$$

$$-\lambda\left.\frac{\partial T}{\partial z}\right|_{z=0^+} + \lambda\left.\frac{\partial T}{\partial z}\right|_{z=0^-} = 2q\,1(t) \tag{32}$$

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Solution:

$$T(t,x,y,z) = T_0 \frac{w}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{b_n b_m}{v_{nm}} \varphi_n\left(\frac{x}{a}\right) \varphi_m\left(\frac{y}{a}\right) \sum_{j=-\infty}^{\infty} \Big[F(u_{1j}, v_{nm}) - F(u_{2j}, v_{nm}) \Big]$$

$$\varphi_n(s) = \sqrt{\frac{2\beta}{\beta + \sin^2 \mu_n}} \cos(\mu_n s)$$

$$F(u,v) = e^{-2uv} \Phi^*(u-v) - e^{2uv} \Phi^*(u+v)$$

$$T_0 = \frac{qa}{\lambda}, \quad \beta = \frac{a\alpha}{\lambda}, \quad w = \frac{\sqrt{kt}}{a}, \quad v_{nm} = w\sqrt{\mu_n^2 + \mu_m^2}, \quad b_n = \varphi_n(0) \frac{\sin(\mu_n)}{\mu_n}$$

$$u_{1j} = \frac{|z + 4Lj|}{2\sqrt{kt}} \qquad u_{2j} = \frac{||z| + 4Lj + 2L_1|}{2\sqrt{kt}}$$

z ... axial space coordinate
L_1 ... length of left sample
L_2 ... length of right sample

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$L = (L_1 + L_2)/2$

x, y ... transversal space coordinates

$2a$ ... transversal size of the sample

q ... heat flow density at source

α ... heat trasfer coefficient for sample - ambient interface

$\Phi^*(u)$ is the complementary error function

μ_n are the roots of equation

$$\beta \cos \mu - \mu \sin \mu = 0$$

END