TEMPERATURE FIELD IN A SOLID BODY RESTRICTED BY TWO PARALEL PLANAR SURFACES

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Abstract

We present the solution of 1D - heat conduction equation using Laplace transformation for a solid body restricted by two parallel planar surfaces which temperatures are constants. Comparison of that solution with measured values of temperature somewhere inside the solid body permits to determine thermal parameters (like diffusivity *a*, thermal conductivityλ, specific heat *c*) of the solid. The formula $a = l^2\theta_c / t_c$ determining diffusivity is derived.

Key words

Isothermal border plane. Temperature field. Determination of thermal parameters.

1. Introduction

Fig. 1 shows a sample the coordinate system and initial and final distribution of the temperature.

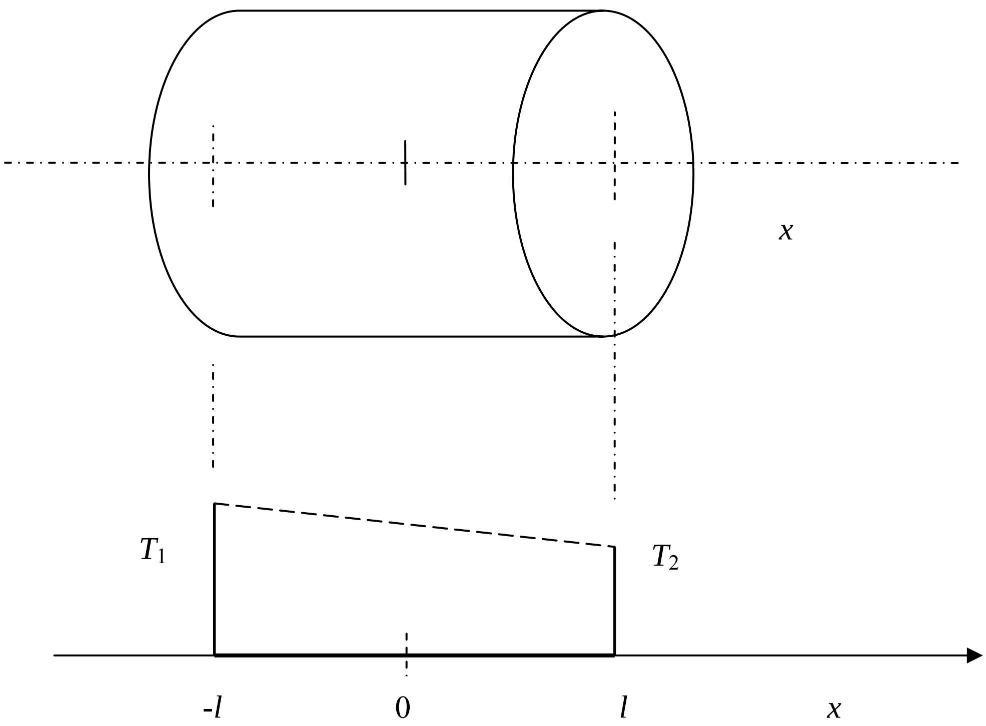

Fig. 1

It is assumed that the temperature at the border surfaces is constant and initial temperature inside a body is zero. After switching on the temperature in time interval $t > 0$ at the border plane $x = -l$ is constant and equals to T_1 as well as at the second border plane $x = l$ where the constant temperature is equal to T_2. In words of mathematics:

$$T\left(x = -l, t\right) = T_1 \qquad t > 0 \tag{1.1}$$

$$T\left(x = l, t\right) = T_2 \qquad t > 0 \tag{1.2}$$

$$T\left(x, t = 0\right) = 0 \qquad -l < x < l, \quad t = 0 \tag{1.3}$$

We are searching for solution $T\left(x,t\right)$ of 1D - heat conduction equation

$$\frac{\partial T}{\partial t} - a\frac{\partial^2 T}{\partial x^2} = 0, \tag{1.4}$$

assuming it obeys boundary (1.1), (1.2) and initial (1.3) conditions. Here $a = \lambda / c\rho$, and ρ is the density of a body.

2. Method of solution

After sufficiently long period of time one can expect that the temperature field inside a body becomes stationary (if $T_1 \neq T_2$). Then

$$\frac{\partial \tau}{\partial t} = 0. \quad \text{In this state} \qquad \frac{d^2\tau}{dx^2} = 0 \tag{2.1}$$

Solution of this equation obeying boundary conditions is

$$\tau\left(x\right) = \frac{T_1\left(l - x\right) + T_2\left(l + x\right)}{2l} \tag{2.2}$$

Solution of non-stationary heat conduction equation we write down as a sum of two functions

$$T\left(x,t\right) = \Theta\left(x,t\right) + \tau\left(x\right) \tag{2.3}$$

Then the boundary conditions for the function.

$$\Theta\left(x,t\right) = T\left(x,t\right) - \tau\left(x\right) \quad \text{are:} \tag{2.4}$$
$$\Theta\left(x = -l,t\right) = T\left(x = -l,t\right) - \tau\left(x = -l\right) = T_1 - T_1 = 0 \tag{2.5}$$
$$\Theta\left(x = l,t\right) = T\left(x = l,t\right) - \tau\left(x = l\right) = T_2 - T_2 = 0 \tag{2.6}$$

and the function Θ fulfils the initial condition

$$\Theta(x,t=0) = T(x,t=0) - \tau(x) = -\tau(x) \tag{2.7}$$

It is easily to show that the Θ function is also a solution of the heat conduction equation

$$\frac{\partial \Theta}{\partial t} - a\frac{\partial^2 \Theta}{\partial x^2} = 0 \tag{2.8}$$

(see Appendix).

We have obtained the solution given by the formula (A.16a) in Appendix. Using that solution and introducing new dimensionless variables: time $\theta = \frac{at}{l^2}$ - (this variable gives the time value t in the new time unit characteristic for a given sample - namely $\left[l^2/a\right]$) - and the space coordinate $\xi = \frac{x}{l}$, $-1 < \xi < 1$. Then considering (A.16a) relative temperature can be expressed as follows

$$\frac{T(\xi,t)}{\tau(\xi)} = \frac{1}{\tau(\xi)}\sum_{n=0}^{\infty}(-1)^n\left[T_1\mathrm{erfc}\left(\frac{(2n+1)+\xi}{2\sqrt{\theta}}\right) + T_2\mathrm{erfc}\left(\frac{(2n+1)-\xi}{2\sqrt{\theta}}\right)\right] +$$

$$\frac{(T_1 - T_2)}{\tau(\xi)}\sum_{n=0}^{\infty}\left[\mathrm{erfc}\left(\frac{(4n+3)+\varsigma}{2\sqrt{\theta}}\right) - \mathrm{erfc}\left(\frac{(4n+3)-\xi}{2\sqrt{\theta}}\right)\right] \tag{2.9}$$

where

$$\tau(\xi) = \frac{1}{2}\left(T_1(1-\xi) + T_2(1+\xi)\right) \quad \text{is the local (at point } \xi \text{) steady temperature} \tag{2.10}$$

One can draw 3D graph of this relative temperature function in good approximation accounting few first terms from infinite series.

The relation (2.9) shows that one of the simplest cases occurs when temperature at both border planes is the same $T_1 = T_2$ and one measures the time development of temperature in the middle of the specimen at $x = 0$. In this case dimensionless temperature is

$$\frac{T(\xi=0,\theta)}{T_1} = 2\sum_{n=0}^{\infty}(-1)^n\,\mathrm{erfc}\left(\frac{(2n+1)}{2\sqrt{\theta}}\right) \tag{2.11}$$

In our approach influence of thermal contact at the border plane is neglected and strictly speaking a heat flow exists along x - axis only. Approximately, this flow prevails so that the heat losses trough cylinder jacket (of a cylindrical sample) during time of measurement temperature is negligible.

3. Universal theoretical temperature dependence on dimensionless time

Now, we want to use the local time development of temperature mentioned above to measure diffusivity *a* of a solid body and measurement to execute in relative short period of time.[1]
If one restricts himself by the first six terms of the series in (3.2) [2]

$$\frac{T\left(\xi=0,\theta\right)}{T_1}=2\sum_{n=0}^{\infty}\left(\text{-}1\right)^n \operatorname{erfc}\left(\frac{\left(2n+1\right)}{2\sqrt{\theta}}\right)\doteq$$
$$f\left(\theta\right)=2\operatorname{erfc}\left(\frac{0.5}{\sqrt{\theta}}\right)-2\operatorname{erfc}\left(\frac{1.5}{\sqrt{\theta}}\right)+2\operatorname{erfc}\left(\frac{2.5}{\sqrt{\theta}}\right)-2\operatorname{erfc}\left(\frac{3.5}{\sqrt{\theta}}\right)+2\operatorname{erfc}\left(\frac{4.5}{\sqrt{\theta}}\right)-2\operatorname{erfc}\left(\frac{5.5}{\sqrt{\theta}}\right) \tag{3.1}$$

and draws a graph of $f\left(\theta\right)$ function (in program Mathematica 4) then can see that it rises relatively steep in an interval $0 < \theta < 2.5$ and approaches close to the equilibrium value of the temperature T_1. This curve does not depend either on dimension of the sample or on diffusivity *a*.

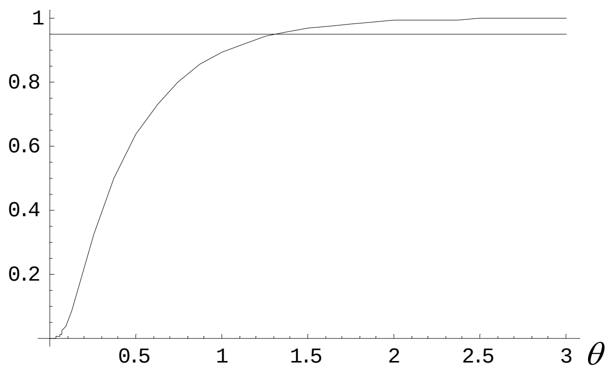

Fig. 2

The dimensionless time variable was introduced as $\theta = at/l^2$. It was already said that quantity l^2/a can be taken for a proper "unit of the tine" of the particular sample. If we would dispose a sample for which this quantity would be $l^2/a=1$ then θ would be equal to *t*, $\theta = t$.

That quantity l^2/a is characteristic for transition of the particular sample into the equilibrium state with temperature T_1. The order of l^2/a is approximately equal to "the time of transition" $t_r \sim l^2/a$.

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[1] One can examine behavior of derivative log *T* with respect to diffusivity *a* (like in [3]) which characterizes sensibility of temperature to a change of diffusivity.

[2] Behavior of a member of this infinite series is discussed in [1] and [2].

The temperature T is less then or equal to T_1. If we choose the value of $T(\theta)$ then the ratio

$$\frac{T(\theta)}{T_1} \doteq f(\theta) \tag{3.2}$$

is known.
The eq. (3.2) determines coordinate θ_c of the intersection point which coordinates are

$$\left[\theta_c, \ \frac{T}{T_1}\right] \qquad \text{(see Fig.2)} \tag{3.3}$$

For example coordinates of the intersection point of that line $T/T_1 = 0.95$ with the curve (3.1) are $[\theta_c, T/T_1] = [\ 1.314,\ 0.95]$. It holds

$$\theta_c = \frac{a}{l^2} t = \frac{\lambda}{c\rho l^2} t \tag{3.4}$$

In the relation (3.4) the values of θ_c and l are known but the diffusivity a and corresponding time t are unknown. So we need to have one more formula connecting these four quantities. It follows; we shall be searching for this formula.

4. Determination of diffusivity using experimental local time development of temperature

Let us assume that temperature dependence on time t at $x = 0$ is known from experiment

$$\frac{T(x=0,t)}{T_1} = f_{\exp}(t) \tag{4.1}$$

Parameters a and l do not appear in the function $f_{\exp}(t)$ explicitly. If this experimental curve will coincide nearly with theoretical (3.1) (after substitution $\theta = at/l^2$ in it), in an interval of time <0, t_m> then one can find from experiment the value t_c and using the formula (3.4) calculate the diffusivity a (or thermal conductivity λ assuming $\rho c l^2$ is known). Then - with respect to (3.4) – it holds

$$a = l^2 \frac{\theta_r}{t_r} \tag{4.2}$$

From Vretenar`s measurement [3] on SiC sample which length was $2l$ = 2.84 mm, we take values of $a = 21.2 \times 10^{-6}\,\mathrm{m}^2 s^{-1}$ and thermal conductivity $\lambda = ac\rho = 46.187$ Wm^{-1}K^{-1} (where the values $\rho = 3242$ kg.m^{-3} and $c = 672$ J.m^{-1}.K^{-1} were accounted). We do not dispose of this experimental curve $f_{\exp}(t)$. Nevertheless; we try to show that above presented method of determination diffusivity really works in this case.

Using some of these data we are going *to test* our approach to determine *a*. We put again $T/T_1 = f_{\exp}(t) = 0.95$. Then that eq. determines the time coordinate t_c.

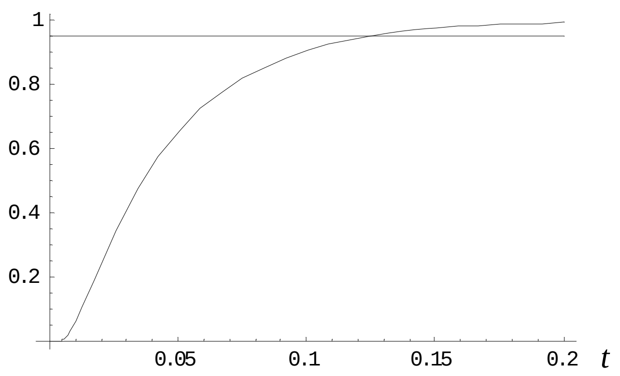

Fig.3

Fig. 4 depicts dependence of $T(x=0,t)/T_1$ given by (3.1) on the time (in seconds). This curve we have obtained by inserting $\theta = \frac{at}{l^2} = \frac{21.2}{1.42^2}t$ in (3.1). That curve supplies experimental one $f_{\exp}(t)$. We are looking for such value of the time *t* for which value of "experimental function" is equal to 0.95 and that value equals to $t_c = 0.125$ s. (It is very short time to gain experimental data needed for determination of temperature development in time.). Then we have obtained the value of diffusivity

$$a = l^2 \frac{\theta_r}{t_r} = (1.42)^2 \times 10^{-6} \frac{1.314}{0.125} \text{ m}^2\text{s}^{-1} = 21.196 \times 10^{-6} \text{m}^2\text{s}^{-1}$$

This is in excellent agreement with Vretenar`s result
It seems to us that more suitable should be a longer sample. For ten times longer sample 2.84 cm the time t_c should be hundred times bigger. In such a case it would be 12.5 s.
Now, we shall discuss shortly the case when $T_2 = 0$ and $\xi = 0$ then

$$\frac{T(\xi=0,\theta)}{\tau(\xi=0)} = \frac{T(\xi=0,\theta)}{T_1/2} = 2\sum_{n=0}^{\infty}(-1)^n \operatorname{erfc}\left(\frac{(2n+1)}{2\sqrt{\theta}}\right) \quad (5.3)$$

This formula shows that temperature approaches its stationary (maximal) value $T_1/2$ in the plane at $x=0$ in the same time interval as before (in the case when T_1 was equal to T_2). But at that time the maximal value at $x=0$ was T_1 - twice as big as now.)
The more detailed analysis of the temperature field given by (2.9) offers further possibilities for determining thermal parameters (see [3]).

5. Conclusions

It is assumed that the time development of the temperature at the center plane at $x = 0$ of a specimen is measured and consequently known. Then it is possible to draw a graph of temperature dependency on the time $T\left(x=0,t\right)=T_1 f_{\exp}\left(t\right)$ in a short time interval $\left\langle 0,t_m \right\rangle$ (t_m represents a few ten seconds). t_c represents the time lying in an interval $\left\langle 0,t_m \right\rangle$.

On the other hand the theoretical time dependence of the temperature is given by (3.1) (in the same time interval). Then it is possible to find the value of θ_c at which $T/T_1 = f\left(\theta_c\right) = f_{\exp}\left(t_c\right)$ holds. The diffusivity value of the sample is expressed by the relation (4.2). We expect that by insertion this value of a as well as l in theoretical temperature dependence (3.1) on time t it will coincide with experimental dependence $f_{\exp}\left(t\right)$.

APPENDIX

Indeed $\quad \dfrac{\partial\left(\Theta+\tau\right)}{\partial t} - a\dfrac{\partial^2\left(\Theta+\tau\right)}{\partial x^2} = 0 \quad$, and $\quad \dfrac{\partial \tau}{\partial\, t} = \dfrac{\partial^2 \tau}{\partial x^2} = 0$

Laplace transformation of the Θ function $\vartheta\left(x,s\right) = L\left[\Theta\left(x,t\right)\right]$ fulfils the eq.

$$L\left\{\frac{\partial \Theta}{\partial t}\right\} - a\frac{\partial^2 L\{\Theta\}}{\partial x^2} = s\vartheta - \left(-\tau\left(x\right)\right) - a\frac{\partial^2 \vartheta}{\partial x^2} = 0$$

This is ordinary nonhomogeneous differential eq. of second order

$$\frac{d^2\vartheta}{dx^2} - \frac{s}{a}\vartheta = \frac{\tau\left(x\right)}{a} \tag{A.1}$$

Further we shall denote $\sqrt{s/a} = k$. Then, the general solution of this eq. is equal to the sum of the general solution

$$\vartheta_0 = A_0 \exp\left(kx\right) + B_0 \exp\left(-kx\right) \tag{A.2}$$

of the corresponding homogeneous eq. $\vartheta'' - k^2\vartheta = 0$ and of some particular solution $\tilde{\vartheta}$ of the nonhomogeneous eq. $\tilde{\vartheta}'' - k^2\tilde{\vartheta} = \tau/a$.

To find particular solution $\tilde{\vartheta}$ the method of variation of parameters A, B was used

$$\tilde{\vartheta} = A\left(x\right)\exp\left(kx\right) + B\left(x\right)\exp\left(-kx\right) \tag{A.3}$$

This leads to expressions

$$A(x) = -\frac{1}{2ak^3}\exp(-kx)\left(k\tau + \frac{T_2 - T_1}{2l}\right)$$

$$B(x) = -\frac{1}{2ak^3}\exp(kx)\left(\left(k\tau - \frac{T_2 - T_1}{2l}\right)\right)$$

Then

$$\tilde{\vartheta} = -\frac{\tau(x)}{ak^2} \tag{A.4}$$

We have obtained the general solution of the nonhomogeneous eq. in form

$$\vartheta(x,s) = A_0\exp(kx) + B_0\exp(-kx) - \frac{T_1(l-x) + T_2(l+x)}{2alk^2}, \quad k = \sqrt{s/a} \tag{A.5}$$

This solution should satisfy transformed boundary conditions

$$\vartheta(x=-l,s) = 0 \tag{A.6}$$

$$\vartheta(x=l,s) = 0 \tag{A.7}$$

Thus, we obtain two eqs. for determination A_0, B_0

$$A_0\exp(-kl) + B_0\exp(kl) - \frac{T_1}{ak^2} = 0 \tag{A.8}$$

$$A_0\exp(kl) + B_0\exp(-kl) - \frac{T_2}{ak^2} = 0 \tag{A.9}$$

witch solution is

$$A_0 = \frac{\exp(-3kl)\left[T_2\exp(2kl) - T_1\right]}{ak^2\left[1 - \exp(-4kl)\right]}, \tag{A.10}$$

$$B_0 = \frac{\exp(-3kl)\left[T_1\exp(2kl) - T_2\right]}{ak^2\left[1 - \exp(-4kl)\right]} \tag{A.11}$$

Thus, the transformed solution satisfying transformed boundary conditions is

$$\vartheta(x,s) = A_0\exp(kx) + B_0\exp(-kx) - \left[T_1(l-x) + T_2(l+x)\right]/2alk^2 =$$

$$\frac{\left[T_2 - T_1\exp(-2kl)\right]\exp\left(-k(l-x)\right) + \left[T_1 - T_2\exp(2kl)\right]\exp\left(-k(l+x)\right)}{ak^2\left[1 - \exp(-4kl)\right]}$$

$$-\left[T_1(l-x) + T_2(l+x)\right]/2alk^2 \tag{A.12}$$

Using the expansion

$$\frac{1}{\left[1-\exp\left(-4kl\right)\right]}=\sum_{0}^{\infty}\exp\left(-4nkl\right)\quad \text{valid for } \; 1>\exp\left(-4kl\right) \tag{A.13}$$

We can rewrite $\vartheta\left(x,s\right)$ as

$$\begin{aligned}\vartheta\left(x,s\right)=&\frac{1}{s}\sum_{0}^{\infty}\left[T_2\exp\left(-\left(4nl+l-x\right)\right)\sqrt{s/a}-T_1\exp\left(-\left(4nl+3l-x\right)\right)\sqrt{s/a}\right]+\\&\frac{1}{s}\sum_{0}^{\infty}\left[T_1\exp\left(-\left(4nl+l+x\right)\right)\sqrt{s/a}-T_2\exp\left(-\left(4nl+3l+x\right)\right)\sqrt{s/a}\right]-\\&\frac{1}{2ls}\left[T_1\left(l-x\right)+T_2\left(l+x\right)\right],\end{aligned} \tag{A.14}$$

One can obtain the inverse Laplace transformation using following formulae $L^{-1}\left[\vartheta\right]=\Theta\left(x,t\right)$

$$L^{-1}\left[\frac{\exp\left(-u\sqrt{s/a}\right)}{s}\right]=\operatorname{erfc}\left(\frac{u}{2\sqrt{at}}\right),\quad \text{holding for} \quad \kappa=\frac{u}{\sqrt{a}}\geq 0 \tag{A.15}$$

(see [1]) and

$$L^{-1}\left[\frac{1}{s}\right]=1,\quad L^{-1}\left[\frac{1}{2ls}\left[T_1\left(l-x\right)+T_2\left(l+x\right)\right]\right]=\frac{1}{2l}\left[T_1\left(l-x\right)+T_2\left(l+x\right)\right]=\tau\left(x\right)$$

it follows

$$\begin{aligned}T\left(x,t\right)=&\sum_{0}^{\infty}\left[T_1\operatorname{erfc}\left(\frac{\left(4n+1\right)l+x}{2\sqrt{at}}\right)+T_2\operatorname{erfc}\left(\frac{\left(4n+1\right)l-x}{2\sqrt{at}}\right)\right]\\&-\sum_{0}^{\infty}\left[T_1\operatorname{erfc}\left(\frac{\left(4n+3\right)l-x}{2\sqrt{at}}\right)+T_2\operatorname{erfc}\left(\frac{\left(4n+3\right)l+x}{2\sqrt{at}}\right)\right]\end{aligned} \tag{A.16}$$

One can see that by the function $T\left(x,t\right)$ the boundary conditions as well as the initial condition are both fulfilled.
The last expression can be rewritten into the form

$$T\left(x,t\right)=\sum_{n=0}^{\infty}(-1)^n\left[T_1\operatorname{erfc}\left(\frac{\left(2n+1\right)l+x}{2\sqrt{at}}\right)+T_2\operatorname{erfc}\left(\frac{\left(2n+1\right)l-x}{2\sqrt{at}}\right)\right]+$$

$$\left(T_1 - T_2\right)\sum_{n=0}^{\infty}\left[\operatorname{erfc}\left(\frac{\left(4n+3\right)l+x}{2\sqrt{at}}\right)-\operatorname{erfc}\left(\frac{\left(4n+3\right)l-x}{2\sqrt{at}}\right)\right] \tag{A.16a}$$

Notice

In [2] p.101, formula (3.9) is introduced (without derivations) which gives the temperature distribution in a slab $-l < x < l$ with constant initial temperature $V_0 \neq 0$ and temperature at border planes maintain zero (Fig. 4)

$$v = V_0 - V_0\sum_{n=0}^{\infty}\left(-1\right)^n\left[\operatorname{erfc}\left(\frac{\left(2n+1\right)l-x}{2\sqrt{at}}\right)+\operatorname{erfc}\left(\frac{\left(2n+1\right)l+x}{2\sqrt{at}}\right)\right] \qquad \text{[2] p.101, f (3.9)}$$

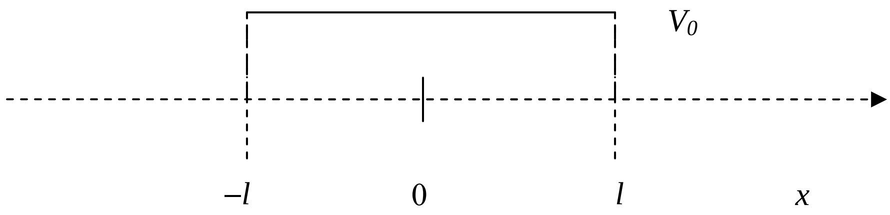

Fig. 4

We show that this is one special case of our formula derived above when $T_1 = T_2$.
If we change V_0 for $\rightarrow -T_1 < 0$ we obtain initial temperature well instead of initial temperature barrier and then we shift the zero temperature down at $-T_1$ we finally obtain

$$T\left(x,t\right) = v + T_1 = T_1\sum_{n=0}^{\infty}\left(-1\right)^n\left[\operatorname{erfc}\left(\frac{\left(2n+1\right)l-x}{2\sqrt{at}}\right)+\operatorname{erfc}\left(\frac{\left(2n+1\right)l+x}{2\sqrt{at}}\right)\right] \tag{A.17}$$

References

[1] Lykov A.V., Teorija teploprovodnosti p.. 585, formula 50 – in Russian (Heat conduction theory))

[2] Carslaw H. S., Jaeger J. C.: Conduction of Heat in Solids, 2nd ed, Oxford University Press, reprinted 1980

[3] Vretenár V.: Extended version of Pulse transient method, Proceedings of Thermo physics, 2005, Contents, pp. 92-103, October 2005, Meeting of the Thermo physical Society.