SIMULATION OF HEAT TRANSPORT EQUATION USING ANSYS SOFTWARE

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Abstract:
This paper thinks over the heat transport problem in plexi-glass environment. We applied the finite element method (FEM) for simple one dimensional example and offer exact description of FEM. Further it shows comparison of experimental data from measurement and data from numerical simulation using software ANSYS, in which the accordance with measurement is demonstrate.

Keywords:
Simulation, finite element method, heat transport

THEORY

Temperature and its spatial distribution represent the driving potential for heat transport. We consider convection and conduction as a process of heat transport. The heat flux density equivalent to conduction we express in equation

$$q = -\lambda . \frac{\partial T}{\partial x}, \tag{1}$$

where λ is thermal conductivity , T temperature and q heat flux density.
The heat flux density due to convection can be written as follows:

$$q = \alpha \left(T_s - T_p \right), \tag{2}$$

where α is heat transfer coefficient, T_p represents environment temperature and T_s surface temperature of the sample, which is in contact with surrounding environment.
The heat transport equation without source term is

$$\frac{\partial T}{\partial t} = \frac{1}{c_p \rho} \frac{\partial}{\partial x} \left(\lambda \frac{\partial T}{\partial x} \right), \tag{3}$$

where T is time and space dependent temperature field, c_p is specific heat capacity and ρ material density.

SOLVED PROBLEM

The aim of our work was the comparison of measured data and simulation, in which we take into consideration all conditions from the experiment (Fig.1). The sample, on which we realized the measurement has shape of cylinder, made of plexi-glass, with diameter 0.03m and length 0.30m (material parameters: $\lambda = 0.195\frac{W}{mK}$, $\rho = 1180\frac{kg}{m^3}$, $c_p = 1465\frac{J}{kgK}$).

Bottom of the cylinder presents a heat source, in direct contact with plexi-glass. Heat source was supplied by direct currant I=0.103A and voltage U=5.24V. Heat flux density from the source has constant value $q = 382\frac{W}{m^2}$ during all measurement (1. boundary condition). Opposite side of cylinder bottom was in contact with ambient air. Its temperature has enduring value T_p=26°C (2.boundary condition) and $\alpha = 10\frac{W}{m^2K}$. The shell of the cylinder was isolated by polystyrene of 5cm thickness, hence the flux through its surface was nought (3.boundary condition). Internal temperature of cylinder was uniform T_o=26°C at the beginning of experiment (initial condition). From previous result, that we can consider only one dimensional problem.

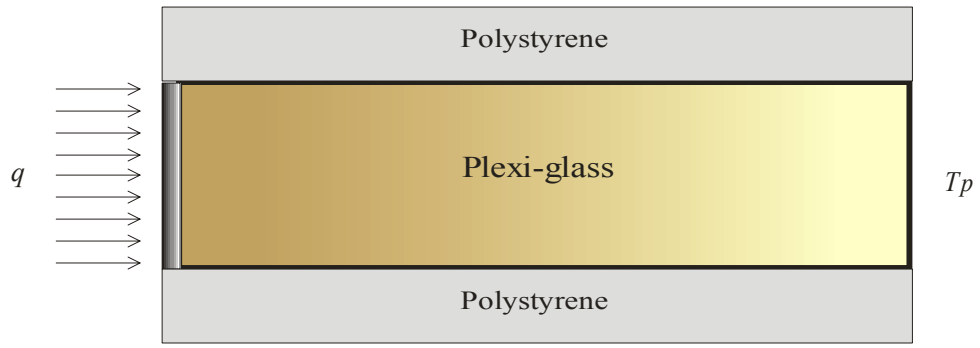


Fig.1

PROBLEM SOLUTION USING FINITE ELEMENT METHOD

We are finding solution u (note $u \equiv T$), which must satisfy mentioned initial and boundary conditions and also heat transport equation, where $c = c_p\rho$ and $a = \lambda$.

$$c\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(a\frac{\partial u}{\partial x}\right) \tag{4}$$

Discretization of domain

All domain, on which are we solving the problem, we divide to n elements and $n+1$ nodes (Fig.2). First node represents surface of plexi-glass next to heat source, which ensure constant heat

flux density inward the sample. Last node is surface on opposite side, which is in contact with surrounding environment. Length of chosen element is h_e.

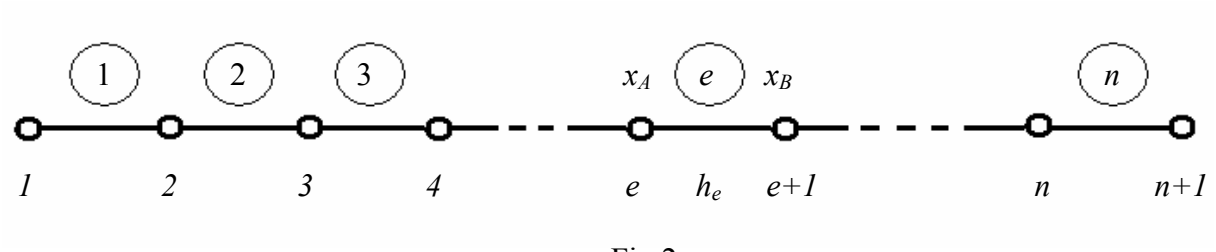


Fig.2

DERIVATION OF ELEMENT EQUATIONS

Derivation of weak form

At first, we multiply the heat transport equation by weight function w. After that we integrate all expression over an element e.

$$\int_{x_A}^{x_B} c\frac{\partial u}{\partial t} w dx - \int_{x_A}^{x_B} \frac{\partial}{\partial x}\left(a\frac{\partial u}{\partial x}\right) w dx = 0 \tag{6}$$

Using Per-partes method we obtain

$$\int_{x_A}^{x_B} c\frac{\partial u}{\partial t} w dx - \left[a\frac{\partial u}{\partial x}\right]_{x_B} w(x_B) + \left[a\frac{\partial u}{\partial x}\right]_{x_A} w(x_A) + \int_{x_A}^{x_B} a\frac{\partial u}{\partial x}\frac{\partial w}{\partial x} dx = 0\,. \tag{7}$$

Fluxes into an element e in positions x_A a x_B are defined as:

$$-a\left[\frac{\partial u}{\partial x}\right]_{x=x_A} = Q_A$$

$$a\left[\frac{\partial u}{\partial x}\right]_{x=x_B} = Q_B$$

Weak form (WF) of transient problem:

$$\int_{\mathbf{x}_A}^{x_B} c\frac{\partial u}{\partial t} dx + \int_{x_A}^{x_B} a\frac{\partial u}{\partial x}\frac{\partial w}{\partial x} dx = Q_B w(x_B) + Q_A w(x_A) \tag{8}$$

Derivation of approximate function

We are finding an approximate solution of transient problem over element e in the shape:

$$u^e(x,t) = \sum_{j=1}^{n} u_j^e(t)\varphi_j^e(x)\,, \tag{9}$$

where time development of the function u is included in time dependent value in j-th nodal point $u_j^e(t)$ and $\varphi_j^e(x)$ is spatial dependent j-th base function (n is number of nodes in element e).

Bases functions must satisfy following conditions. They have to be continuous, differentiable (up to the order, which requires the weak form) and values of approximate solution in node points are equal to real solution.

For example, the bases functions for linear approximation of solution u ($u^e = ax + b$) over element have shapes:

$$\varphi_1^e = \frac{x_B - x}{x_B - x_A}, \qquad \varphi_2^e = \frac{x - x_A}{x_B - x_A}$$

Derivation of element equations

Spatial semidiscretization

Instead of u in WF we substitute approximate solution u^e and as a weight functions w we use bases functions φ_i.

$$\int_{x_A}^{x_B} c \frac{\partial\left(\sum_{j=1}^{n} u_j^e(t)\varphi_j^e(x)\right)}{\partial t}\varphi_i^e(x)dx + \int_{x_A}^{x_B} a \frac{\partial\left(\sum_{j=1}^{n} u_j^e(t)\varphi_j^e(x)\right)}{\partial x}\frac{\partial\varphi_i^e(x)}{\partial x}dx = Q_A\varphi_i^e(x_A) + Q_B\varphi_i^e(x_B) \quad (10)$$

Now we have the system of differential equations over element e

$$\sum_{j=1}^{n}\frac{\partial u_j^e(t)}{\partial t}\int_{x_A}^{x_B} c\varphi_j^e(x)\varphi_i^e(x)dx + \sum_{j=1}^{n} u_j^e(t)\int_{x_A}^{x_B} a\frac{\partial\varphi_j^e(x)}{\partial x}\frac{\partial\varphi_i^e(x)}{\partial x}dx = Q_A\varphi_i^e(x_A) + Q_B\varphi_i^e(x_B) = F_i, \quad (11)$$

where $M_{ij} = \int_{x_A}^{x_B} c\varphi_j^e(x)\varphi_i^e(x)dx$ is mass matrix and $K_{ij} = \int_{x_A}^{x_B} a\frac{\partial\varphi_j^e(x)}{\partial x}\frac{\partial\varphi_i^e(x)}{\partial x}dx$ is stiffness matrix.

For linear approximation of solution the matrices have such shape $M = ch_e\begin{pmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{pmatrix}$, $K = \frac{a}{h^e}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

Further we introduce vectors: $\vec{u} = \left(u_1^e(t),\ u_2^e(t),\ ....,\ u_n^e(t)\right)$

$$\dot{\vec{u}} = \left(\dot{u}_1^e(t),\ \dot{u}_2^e(t),\ ....,\ \dot{u}_n^e(t)\right)$$

Then we get simplified matrix notation:

$$M.\dot{\vec{u}} + K.\vec{u} = \vec{F} \quad (12)$$

Time discretization

Let us consider the time evolution of $u_j^e(t)$ at j-th node of element e at time t_s with timestep Δt according to Fig. 3.

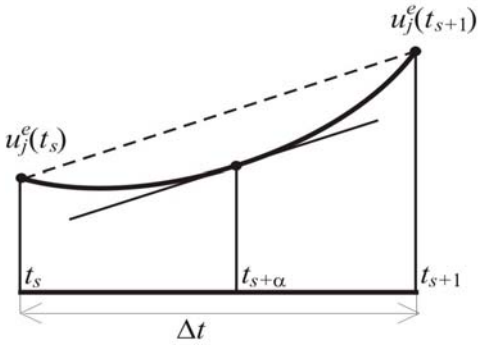


Fig.3

The value of $u_j^e(t_s)$ raises during the timestep Δt to $u_j^e(t_{s+1})$. This new value can be expressed as the sum of function value at time t_s and its increment during timestep Δt. The difficulty is in accuracy of calculating the increment. One possibility is to take the rate of temperature increase from the derivation at time t_s (we take the Taylor's expansion at time t_s and use the first two terms of it to calculate the value at time t_{s+1}). As we can see from the curve describing the development of temperature depending on time this does not fit the real dependency and the following calculations would be signate by this inaccuracy. This is because the derivative has to be consistent with the slope of line (dash line) which connecting values $u_j^e(t_s)$ and $u_j^e(t_{s+1})$. Therefore we look for another way to determine the derivative value. It is clear that there is always a point during the time step Δt where the derivative of function $u_j^e(t)$ is consistent with the tangent of the dash line. We denote it $t_{s+\alpha}$ (α can reach the value within 0 to 1). It is difficult to estimate the correct value of α. However the value of the derivative of function $u_j^e(t)$ at arbitrary chosen time during the timestep lies between of the derivative values at times t_s and t_{s+1}, therefore we achieve more accuracy. We replace the derivative at time t_s by the derivative at time $t_{s+\alpha}$ in the Taylor's expansion which results in reducing the error caused by neglecting the higher terms of the expansion. We calculate $\dot{u}_j^e(t_{s+\alpha})$ from weighted average of the derivatives at times t_s and t_{s+1}.
Then we can write:

$$u_j^e(t_{s+1}) = u_j^e(t_s) + \Delta t.\dot{u}_j^e(t_s) = u_j^e(t_s) + \Delta t.\dot{u}_j^e(t_{s+\alpha}) = u_j^e(t_s) + \Delta t.\left(\frac{(1-\alpha).\dot{u}_j^e(t_s) + \alpha.\dot{u}_j^e(t_{s+1})}{(1-\alpha)+\alpha}\right) \quad (13)$$

When we consider $\vec{u}_s = \vec{u}(t_s)$
$\vec{u}_{s+1} = \vec{u}(t_{s+1})$

then

$$\vec{u}_{s+1} = \vec{u}_s + \Delta t.\left((1-\alpha)\dot{\vec{u}}_s + \alpha.\dot{\vec{u}}_{s+1}\right). \quad (14)$$

We multiply WF in time t_s ($M.\dot{\vec{u}}_s + K.\vec{u}_s = \vec{F}_s$) by the term $\Delta t.(1-\alpha)$ and WF formulation in time t_{s+1} ($M.\dot{\vec{u}}_{s+1} + K.\vec{u}_{s+1} = \vec{F}_{s+1}$) by the term $\Delta t\alpha$. Afterward we sum both equations. Then we obtain an equation without time derivations.

$$M\left(\Delta t(1-\alpha).\dot{\vec{u}}_s + \Delta t\alpha.\dot{\vec{u}}_{s+1}\right) + \Delta t(1-\alpha)K.\vec{u}_s + \Delta t\alpha.K.\vec{u}_{s+1} = \Delta t\left((1-\alpha)\vec{F}_s + \alpha.\vec{F}_{s+1}\right) \quad (15)$$

$$M\left(\vec{u}_{s+1} - \vec{u}_s\right) + \Delta t(1-\alpha)K.\vec{u}_s + \Delta t\alpha.K.\vec{u}_{s+1} = \Delta t\left((1-\alpha)\vec{F}_s + \alpha.\vec{F}_{s+1}\right) \quad (16)$$

We put unknowns $\vec{u}_{s+1}$ on the left side and knowns $\vec{u}_s$ on the other side of equation.

$$(M + \Delta t\alpha.K).\vec{u}_{s+1} = M.\vec{u}_s + \Delta t(\alpha-1)K.\vec{u}_s + \Delta t\left((1-\alpha)\vec{F}_s + \alpha.\vec{F}_{s+1}\right) \quad (17)$$

Finally we have got system of algebraic equations over the element.

$$H.\vec{u}_{s+1} = \vec{G}_s \quad (18)$$

For value α we can choose some possibilities:

$\alpha = 0$	explicit Euler method	it can be unstable if we choose wrong time step

$\alpha = 1$	implicit Euler method	stable method
$\alpha = \frac{1}{2}$	Frank-Nicolson method	stable method

Derivation of global finite element model

Further we give some indications how to proceed to derive global system of equations over whole domain.
We consider two adjacent elements *e* and *e+1*. Their assembling is possible if we follow two conditions: 1, continuity of primary variables in connecting node points, it means $u_n^e = u_1^{e+1}$ (For the mesh of linear elements we can write: $u_1^1 = u_1$, $u_2^1 = u_1^2 = u_2$, ... $u_2^{n=1} = u_1^n = u_n$), and 2, balance of secondary variables in connecting nodes, it means $Q_B^e = -Q_A^{e+1}$
If we satisfy these simple rules, the systems of equations over each element will be arranged to one system of global equations, in which the numbering of nodes will be also global.

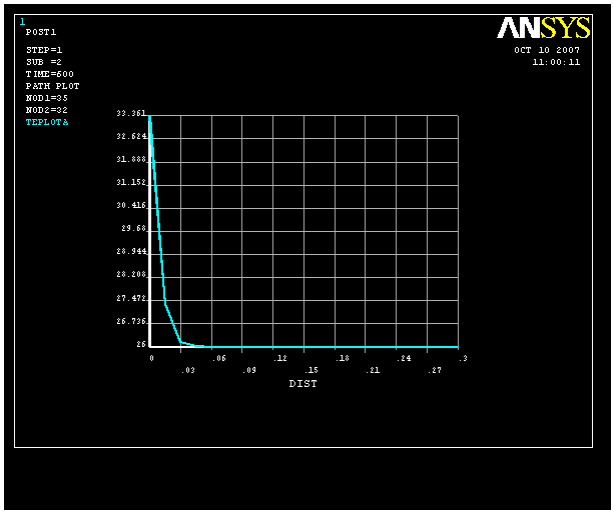
INCLUSION AND DISCUSSION

The flux from the heat source in experiment seems to be constant, as we can consider from constant temperature differences between source and adjacent plexi-glass surface, during experiment. We were also watching the surface temperatures on cylinder shell, that were in far regions from the source, about few tenths degree lower than on axis. In close region the differences were in order of few degrees. It shoves on fact, that dissipation fluxes over cylinder shell are not negligible, so we have to take this information into account in future simulation and instead of Neumann condition we must simulate exact structure of the equipment. Otherwise, the simulation verifies the measured data, as we can see from comparison of the simulation plots (Plot1,…, Plot4) and table of experimental data (Tab.1). We did not plot the dependence of measured data, because of wanting distances between the probes. Thus, for better monitoring of temperature field closer to heat source we have to insert much more probes into this area.

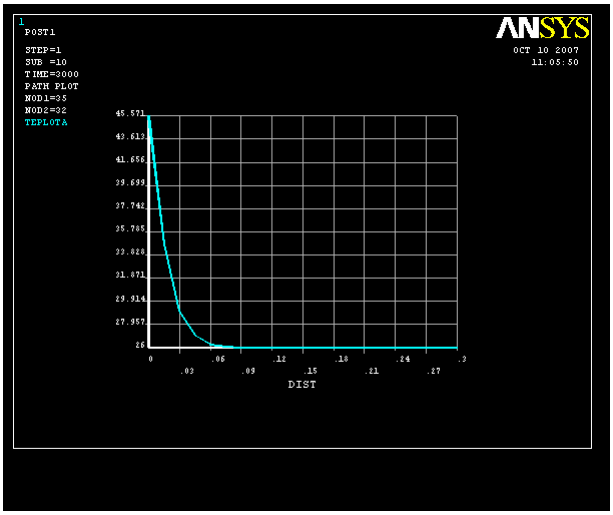
SIMULATION RESULTS AND THEIR COMPARISON WITH EXPERIMENT

Tab.1: Temperatures were measured on the cylinder axis in various distances from the source: 0, 0.1, 0.2 a 0.3m, during time period 0-200min.

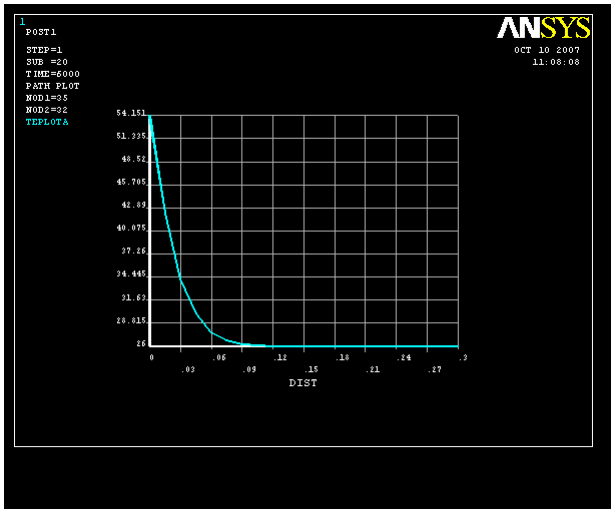
	Heat source	0	0,1	0,2	0,3
0	26,075	25,940	26,100	26,200	26,425
5	35,000	32,680	26,125	26,225	26,400
10	38,600	36,350	26,125	26,225	26,400
20	43,100	40,850	26,150	26,225	26,425
30	46,275	43,875	26,225	26,250	26,425
50	50,300	48,000	26,350	26,250	26,425
100	55,460	53,550	26,800	26,300	26,475
150	57,975	56,025	27,350	26,375	26,500
200	59,075	57,450	27,800	26,450	26,525



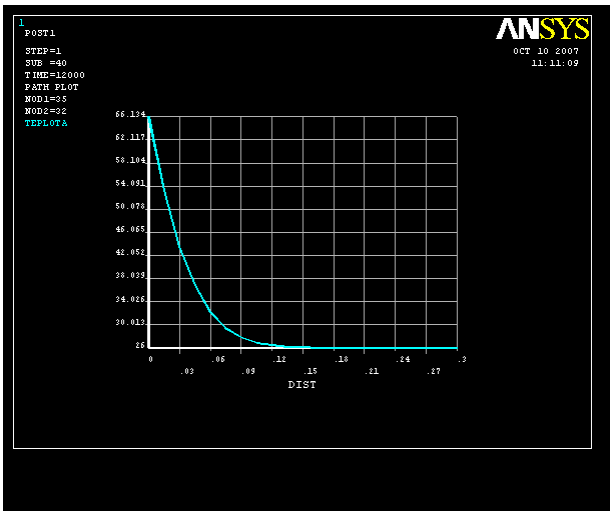
Plot1



Plot2



Plot3



Plot4

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REFERENCES

[1] J. N. Reddy: *An introduction to the finite element method*, Singapore, 2nd Edition, 1993