Derivation of the Boundary Condition for a Heat Source

F. Čulík , J. Lukovičová[1]

Abstract

A special method of derivation the boundary condition is shown in this paper on the base of the inhomogeneous heat conduction equation. The term on the right side of the equation considers the influence of a heat source in creation the corresponding temperature field in solids. The 1-D case for a planar heat source is studied in details.

1. Determination and a physical meaning of the heat source term

1.1 An Introduction

Inhomogeneous differential heat conduction equation can be derived on the base of internal energy balance equation

$$\frac{\partial u(\boldsymbol{r},t)}{\partial t} + \nabla \cdot \boldsymbol{q}(\boldsymbol{r},t) = \sigma_u(\boldsymbol{r},t) \tag{1}$$

where the density of internal energy is $u(\boldsymbol{r},t)$, the heat flux is $\boldsymbol{q}(\boldsymbol{r},t)$ (the density of the heat flow) and $\sigma_u(\boldsymbol{r},t)$ - is a source term. It is assumed that the mass transport does not exist. In a case, when no work is done by the system (or on the system) it is a heat source term according to the first law of thermodynamics.

The equation (1) defines the heat flux $\boldsymbol{q}(\boldsymbol{r},t)$ [1] together with the eqs. (2), (3), (4)

$$\frac{\partial u}{\partial t} = \rho c \frac{\partial T}{\partial t} \tag{2}$$

(ρ, c – density and specific heat)

$$\sigma_u = \frac{d\dot{Q}}{dV} = \frac{dP}{dV} \tag{3}$$

where $\dot{Q} = P$ is the heat power of a heat source (V – volume).

---

[1] František Čulík, Jozefa Lukovičová, Dpt. of Physics, Civil Engineering Faculty, Slovak University of Technology, Bratislava, SR.

$$\boldsymbol{q} = -\lambda \text{ grad } T \tag{4}$$

The eq. (4) is known as the first Fourier's law (λ – thermal conductivity, T – temperature).

2. A planar heat source

Further on, we restrict our study to a planar heat source located at $x = 0$. This is the border of two media in thermal contact: (1) $x < 0$ and (2) $x > 0$ with different thermal properties ρ_1, c_1, λ_1; ρ_2, c_2, λ_2 (density, specific heat, thermal conductivity). We assume that the heat flux

$$\boldsymbol{q} = \left(q\left(x,t\right),0,0\right) \tag{5}$$

and isothermal surfaces are planes orthogonal to x-axis.

Then, the problem is reduced to one-dimensional problem 1-D. In this case the eq. (1) takes the form

$$\rho c \frac{\partial T\left(x,t\right)}{\partial t} + \frac{\partial q\left(x,t\right)}{\partial x} = \frac{1}{S}\frac{\partial P\left(x,t\right)}{\partial x} \tag{6}$$

S is the area of a heat source. Integration of this equation (fig. 1) in limits

$$-\varepsilon \le x \le \varepsilon, \quad \varepsilon \ge 0\,, \quad y,z \in S \tag{7}$$

Fig. 1

---

gives

$$S\int_{-\varepsilon}^{\varepsilon}\left(\rho c \frac{\partial T\left(x,t\right)}{\partial t} + \frac{\partial q\left(x,t\right)}{\partial x}\right)dx = \int_{-\varepsilon}^{\varepsilon}\frac{\partial P\left(x,t\right)}{\partial x}\,dx \tag{8}$$

From this eq. we obtain after a limiting process

$$q_2\left(+0,t\right) - q_1\left(-0,t\right) = q_s\left(t\right) \tag{9}$$

because

$$\lim_{\varepsilon\to 0}\left\{\rho_2 c_2 \int_0^{\varepsilon}\frac{\partial T_2\left(x,t\right)}{\partial t}dx - \rho_1 c_1 \int_0^{-\varepsilon}\frac{\partial T_1\left(x,t\right)}{\partial t}dx\right\} =$$

$$\lim_{\varepsilon\to 0}\left\{\rho_2 c_2 \frac{\partial}{\partial t}\int_0^{\varepsilon}T_2\left(x,t\right)dx - \rho_1 c_1 \frac{\partial}{\partial t}\int_0^{-\varepsilon}T_1\left(x,t\right)dx\right\} = 0 \tag{10}$$

$$\lim_{\varepsilon \to 0} \int_{-\varepsilon}^{\varepsilon} \frac{\partial q(x,t)}{\partial x} dx = \lim_{\varepsilon \to 0} \left[q_2(\varepsilon,t) - q_1(-\varepsilon,t) \right] = q_2(+0,t) - q_1(-0,t) \tag{11}$$

$$\lim_{\varepsilon \to 0} \int_{-\varepsilon}^{\varepsilon} \frac{\partial P(x,t)/S}{\partial x} dx = \lim_{\varepsilon \to 0} \int_{-\varepsilon}^{\varepsilon} q_s(t)\delta(x)\, dx = q_s(t) \tag{12}$$

where

$$q_s(t) = P(t)/S \tag{13}$$

This is in accord with our assumption that a heat source is placed in the plane at $x = 0$
This implies

$$P(x,t)/S = \frac{P(t)}{S}\theta(x) = q_s(t)\theta(x) \tag{14}$$

$\theta(x)$ is the Heaviside's step-wise function

$$\theta(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases} \tag{15}$$

and following relations were used for the Dirac function $\delta(x)$

$$\delta(x) = \frac{d\theta(x)}{dx}, \qquad \int_{-\varepsilon}^{\varepsilon} \delta(x)\, dx = 1 \tag{16}$$

Accounting the Fourier's law the boundary condition (9) can be rewritten as follows

$$\lambda_1 \frac{\partial T_1(-0,t)}{\partial x} - \lambda_2 \frac{\partial T_2(+0,t)}{\partial x} = q_s(t) \tag{17}$$

Integral form of the boundary condition (17) is

$$\lambda_1 \int_0^t \frac{\partial T_1(-0,t')}{\partial x} dt' - \lambda_2 \int_0^t \frac{\partial T_2(+0,t')}{\partial x} dt' = \frac{1}{S}\int_0^t P(t')dt' \tag{18}$$

The heat produced by unit area of the planar heat source in the time interval *t*

$$Q = \int_0^t q_s(t')dt' = \frac{1}{S}\int_0^t P(t')dt' \tag{19}$$

is splitting into two parts: The heat transferred through unit area into medium (1) (to the left) with thermal conductivity λ_1 is equal

$$Q_1 = -\int_0^t q_1\left(-0,t'\right)dt' = \lambda_1 \int_0^t \frac{\partial T_1\left(-0,t'\right)}{\partial x} dt' \tag{20}$$

and that one transferred through unit area into medium (2) (to the right) with thermal conductivity λ_2 equals

$$Q_2 = \int_0^t q_2\left(-0,t'\right)dt' = -\lambda_2 \int_0^t \frac{\partial T_2\left(-0,t'\right)}{\partial x} dt' \tag{21}$$

Now, eq. (18) can be written as

$$Q_1 + Q_2 = Q \tag{22}$$

3. Discussion

The boundary condition (9) says: A planar heat source creates discontinuity of the normal component of the heat flux in the plane of its location. The difference of these components by crossing this plane equals the heat power of a heat source unit area. The heat flow is directed from the planar heat source away (to the media where temperature is lower that one of the source.) This is a heating process of the media.

In a case of cooling when a planar heat source is replaced e.g. by a cold thin plate the direction of the heat flow will be opposite (to the cold plate from media assuming their temperature is higher that one of the cold plate).

Finally, if there is no heat source the normal component remains continuous. Then, it holds at the both sides of the plane $x = 0$: $q_1\left(+0,t\right) = q_2\left(-0,t\right)$

In theoretical determination of the temperature field there is a need to solve differential heat conduction equation under the initial and boundary conditions. Knowledge of temperature development at a given place in a material is necessary for experimental investigation of its thermal parameters. We mention here the Pulse transient method. Mathematical formulae describing a temperature field in a specimen contain thermal parameters. Fig. 2 shows the scheme of experimental setup (in approximation when 3-D case is reduced to 1-D) [2].

Fig. 2

The heat pulse creates a planar heat source at $x = 0$. It is the border plane of two specimen parts $1 \Leftrightarrow 2$. A thermocouple serves for temperature measurement at $x = h$ between $2 \Leftrightarrow 3$. Initial temperature is constant (the same in each of tree parts being in mutual thermal contact). A small heat pulse is applied to the specimen. Unknown thermal parameters of the middle part namely, thermal diffusivity, thermal conductivity and specific heat are then calculated upon temperature response according to a simplified model.

Acknowledgement
Financial support of the project VEGA 1/4204/07 is gratefully acknowledged.

References

[1] H. J. Kreuzer, Nonequilibrium Thermodynamics and its Statistical Foundations, Clarendon Press-Oxford 1981

[2] Vretenár V.: Extended version of Pulse transient method, Proceedings of Thermophysics, 2005, Contents, pp. 92-103, October 2005, Meeting of the Thermophysical Society,_SAS Bratislava 2005. http://www.tpl.ukf.sk/engl_vers/thermophys/

APPENDIX

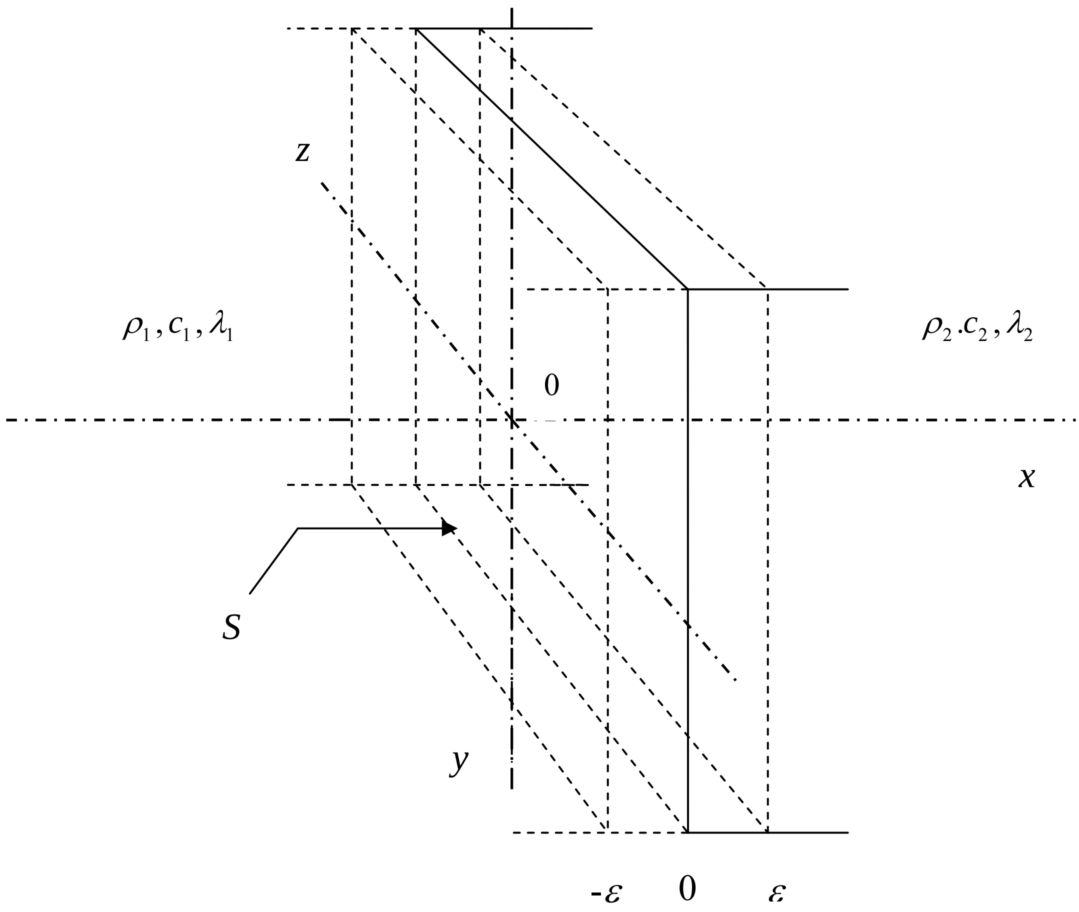

Fig.1. Two layers with different thermal properties each of the thickness ε. A planar heat source acts in the plane at $x = 0$. This is the plane of a mutual thermal contact of both media.

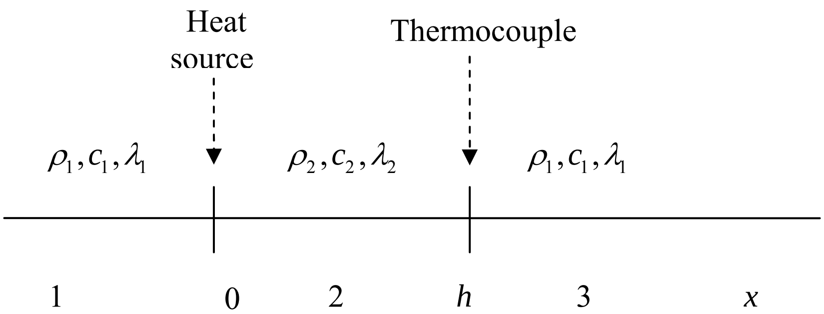

Fig. 2 A specimen setup is consisting of two outer parts (1, 3) hawing known identical thermal properties and one middle part (2) of unknown thermal properties.