Accuracy of transient methods

P. Dieška

Department of Physics, STU FEI

Ilkovičova 3, 812 19 Bratislava, Slovakia

October 11, 2007

Concept

- Experiment $$\{t_n, T_n\}_{n=1}^{N}$$

$$\mathbf{b} = \{b_j\}_{j=1}^{N_b}$$

- Model $$T_{model} = f(t, \mathbf{a}, \mathbf{b})$$

$$\mathbf{a} = \{a_i\}_{i=1}^{N_a}$$

- Fitting model parameters to experimental data: determined, overdetermined problem

Accuracy estimation - overdetermined problem

Least squares optimization $\min\{\sum_{n=1}^{N}[T_n - f_n(\mathbf{a},\mathbf{b})]^2\}$

where $f_n(\mathbf{a},\mathbf{b}) = f(t_n,\mathbf{a},\mathbf{b})$

t_n ... deterministic parameter
$\mathbf{b}$, $\{T_n\}_{n=1}^{N}$ ... independent random variables
$\mathbf{a}$ ... dependent random variables

$$\sum_{n=1}^{N}(T_n - f_n)\frac{\partial f_n}{\partial a_i} = 0, \qquad i = 1, 2, \ldots N_a \tag{1}$$

There are estimated the moments:

$$< a_i > \text{ as a solution of equation (1)}$$
$$< b_j > \text{ from other measurements}$$

$$< (b_j - < b_j >)(b_k - < b_k >) > \sim \delta_{jk} u(b_j)^2 \quad (2)$$
$$< T_n > \sim f_n(< \mathbf{a} >, < \mathbf{b} >) \quad (3)$$
$$< (T_n - < T_n >)(T_m - < T_m >) > \sim \delta_{nm} u(T)^2 \quad (4)$$
$$< (T_n - < T_n >)(b_k - < b_k) > \sim 0 \quad (5)$$

Now we estimate the uncertainty

$$u(a_i)^2 \sim < (a_i - < a_i >)^2 > = < (\Delta a_i)^2 > \sim < (da_i)^2 > \quad (6)$$

Differentiating the equation (1) we obtain

$$\sum_{i=1}^{N_a} A_{ki} da_i = \sum_{n=1}^{N} dT_n \frac{\partial f_n}{\partial a_k} + \sum_{j=1}^{N_b} B_{kj} db_j, \quad k = 1, 2, ..N_a \tag{7}$$

where

$$A_{ki} = \sum_{n=1}^{N} \left[(f_n - T_n) \frac{\partial^2 f_n}{\partial a_k \partial a_i} + \frac{\partial f_n}{\partial a_k} \frac{\partial f_n}{\partial a_i} \right] \simeq \sum_{n=1}^{N} \frac{\partial f_n}{\partial a_k} \frac{\partial f_n}{\partial a_i} \tag{8}$$

$$B_{kj} = -\sum_{n=1}^{N} \left[(f_n - T_n) \frac{\partial^2 f_n}{\partial a_k \partial b_j} + \frac{\partial f_n}{\partial a_k} \frac{\partial f_n}{\partial b_j} \right] \simeq -\sum_{n=1}^{N} \frac{\partial f_n}{\partial a_k} \frac{\partial f_n}{\partial b_j} \tag{9}$$

Solution of equation (7) has the form

$$da_k = \sum_{i=1}^{N_a} A_{ki}^{-1} \Big(\sum_{n=1}^{N} dT_n \frac{\partial f_n}{\partial a_i} + \sum_{j=1}^{N_b} B_{ij} db_j \Big), \quad k = 1, 2, ..N_a \tag{10}$$

and the uncertainty of a_k is with respect to equation (2) - (6) and (10)

$$u(a_k)^2 = \sum_{i=1}^{N_a} \sum_{i'=1}^{N_a} A^{-1}_{ki} A^{-1}_{ki'} \left[A_{i'i} u(T)^2 + \sum_{j=1}^{N_b} \sum_{j=1}^{N_b} B_{ij} B_{i'j} u(b_j)^2 \right] \tag{11}$$

$$\boxed{u(a_k)^2 = C^2_{kT} u(T)^2 + \sum_{j=1}^{N_b} C^2_{kj} u(b_j)^2}$$

where the sensitivities are

$$C_{kT} = \sqrt{A^{-1}_{kk}} \tag{12}$$

and

$$C_{kj} = \sum_{i=1}^{N_a} A^{-1}_{ki} B_{ij} \tag{13}$$

The elements of matrix $\mathbf{A}$ and $\mathbf{B}$ are defined with equations (8) and (9). It is seen that

$$\mathbf{A} \sim N \quad \mathbf{B} \sim N$$

therefore $C_{kT} \sim \frac{1}{\sqrt{N}}$ and C_{kj} is N-independent. For power-like dependence it is useful to define indices

$$\nu_{kj} = \frac{\partial \log(a_k)}{\partial \log(b_j)} = \frac{b_j}{a_k} C_{kj} \tag{14}$$

Then for relative uncertainties we can write the equation

$$u_r(a_k)^2 = C_{kT}^2 \frac{u(T)^2}{a_k^2} + \sum_{j=1}^{N_b} \nu_{kj}^2 u_r(b_j)^2 \tag{15}$$

Application to simple model

1D model: The temperature response on the stepwise heat flow from planar heat source:

$$T(t,x) = T_0\left[\frac{e^{-v^2}}{\sqrt{\pi}v} - \Phi^*(v)\right] \tag{16}$$

where $T_0 = qx/\lambda$
q ... heat flow density at source
x ... axial space coordinate of thermometer
λ ... thermal conductivity
$v = \frac{x}{2\sqrt{kt}}$
k ... thermal diffusivity
t ... time
We apply the formula (13) on the relation (16). The obtained indices are constant ones:

$\nu_{k,x} = 2$	$\nu_{k,q} = 0$
$\nu_{\lambda,x} = 1$	$\nu_{\lambda,q} = 1$

Application to more complicated model

2D model: The temperature response on the stepwise heat flow from planar heat source - cylindric sample with radial Newtonian heat transfer from sample surface:

$$T(t,x) = T_0 \frac{R}{x} \sum_{\xi} \frac{\beta}{\xi(\xi^2+\beta^2) J_0(\xi)} \left[e^{-2uv} \Phi^*(v-u) - e^{2uv} \Phi^*(v+u) \right] \quad (17)$$

where Φ^* is the complementary error function,

$u = \xi \frac{\sqrt{kt}}{R}$

R ... sample radius

$\beta = \frac{R\alpha}{\lambda}$

α ... heat transfer coefficient

ξ is the root of the equation

$$\beta J_0(\xi) - \xi J_1(\xi) = 0$$

J_0, J_1 ... Bessel functions of the first kind

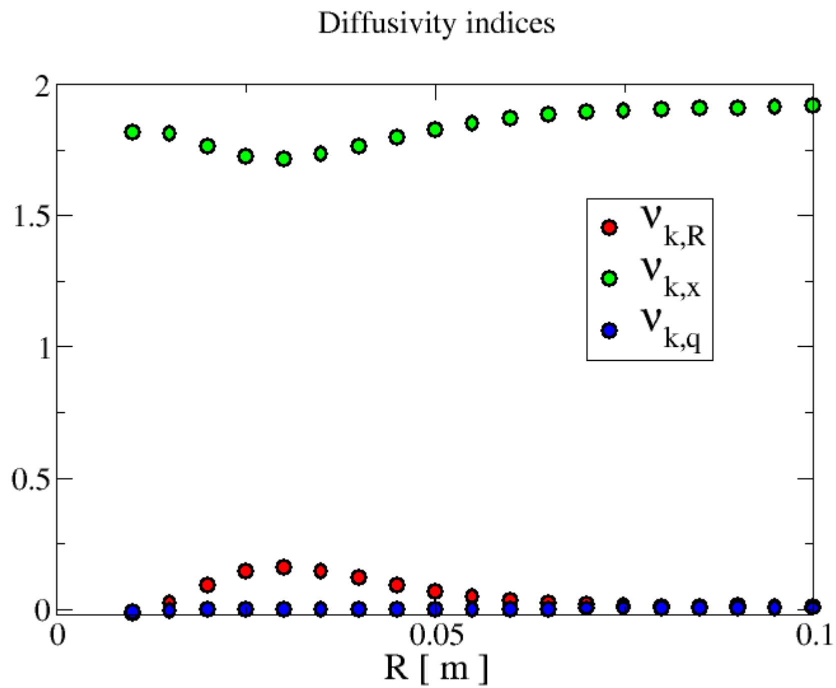
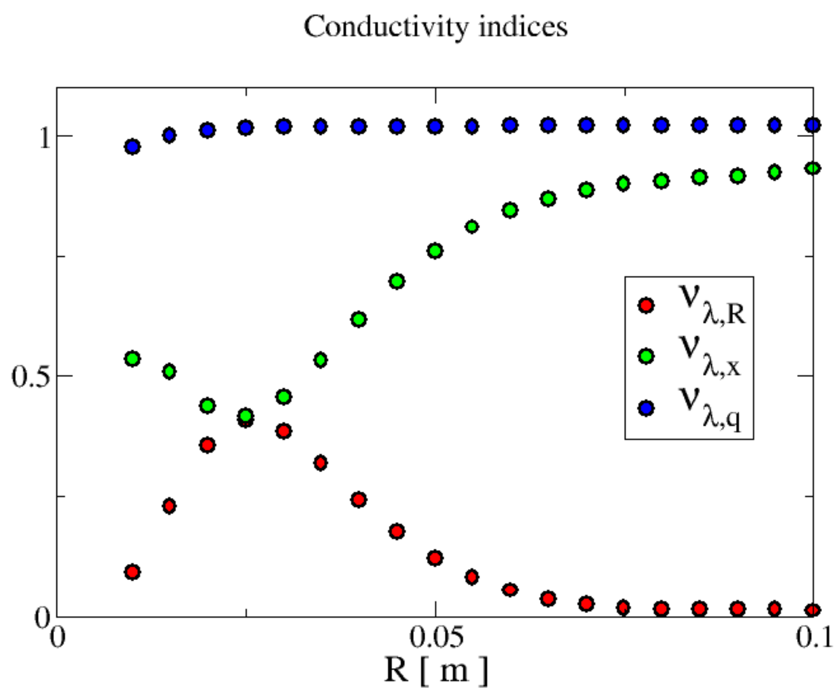


Figure 1:



Conductivity indices
1
0.5
0
0
0.05
0.1
R [m]
$\nu_{\lambda,R}$
$\nu_{\lambda,x}$
$\nu_{\lambda,q}$

Figure 2:

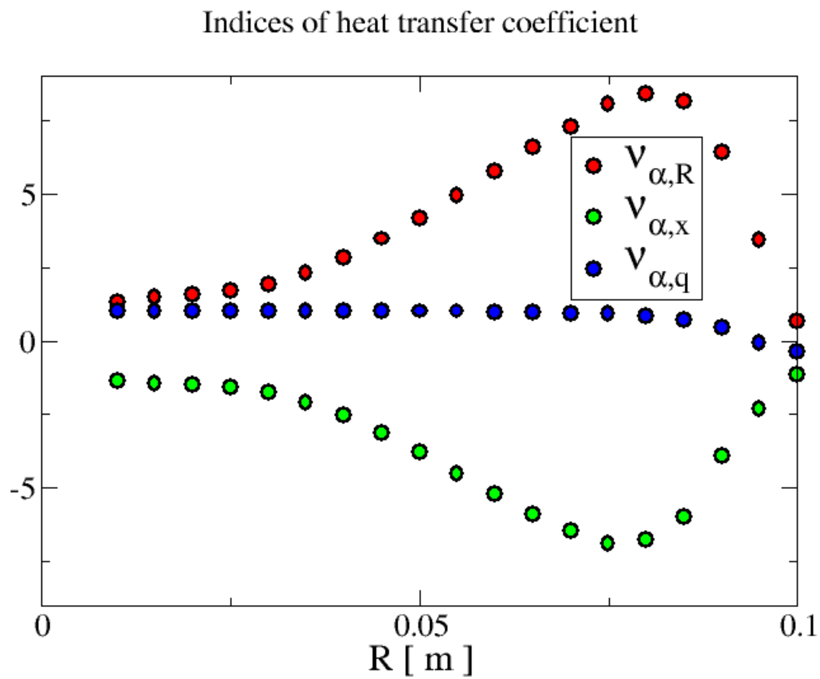

Indices of heat transfer coefficient
5
0
-5
0
0.05
0.1
R [m]
ν α,R
ν α,x
ν α,q

Figure 3: