INVERSE HEAT TRANSFER ANALYSES AS A TOOL OF MATERIAL PARAMETER ESTIMATION

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Abstract

Different *Inverse Heat Transfer Problems* (IHTP) have been formulated and classified. An algorithm for the solution of inverse problems by means of the sensitivity coefficients method has been suggested, applying the simplest – but most effective – objective function. Considerable attention has been paid to the preparation of experiments from numerical (mathematical) point of view, aiming at measurements of temperature at the boundary of the investigated object, to be used later as input data in the inverse analysis, reproducing *e.g.* the boundary conditions or thermal conductivity. Attention has been paid rather to the procedure and less to mathematical formalism. At the end of the paper the most important monographs are quoted for those who are interested in the solution of the theoretical and practical inverse heat transfer problems.

Keywords:

Inverse problems, sensitivity analysis, heat tranfer

1. Inverse Problems of Thermal Conduction

The temperature in a given area $\Omega = \Omega(\mathbf{x}) = \Omega(x, y, z) = \Omega(x_i)$, $(i = 1,2,3)$ is described by the following differential equation:

$$\operatorname{div}\left[k_k(T)\nabla T(\mathbf{x},t)\right] = \rho_k c_k(T)\frac{\partial T(\mathbf{x},t)}{\partial t} + q_{\mathrm{V}} \quad \text{for} \quad k = 1,2,...,N_{\mathrm{k}} \quad \text{and for} \quad \mathbf{x}\in\Omega, t\in(0,\infty) \tag{1}$$

together with the respective boundary and initial conditions:

$$-k(T)\frac{\partial T(\mathbf{x},t)}{\partial n} = q_i \quad \text{for} \quad i = 1,2,...,N_{\mathrm{q}} \quad \text{and} \quad \text{for} \quad \mathbf{x}\in\partial\Omega_i, t\in(0,\infty) \tag{2}$$

$$-k(T)\frac{\partial T(\mathbf{x},t)}{\partial n} = h_j\left(T(\mathbf{x},t) - T_{\mathrm{a}}\right) \quad \text{for} \quad j = 1,2,...,N_{\mathrm{h}} \quad \text{and} \quad \text{for} \quad \mathbf{x}\in\partial\Omega_j, t\in(0,\infty) \tag{3}$$

$$T(\mathbf{x},0) = T_0(\mathbf{x}) \quad \textit{for} \quad \mathbf{x}\in\Omega \tag{4}$$

This means that if the causes are known, *viz.*

- the heat fluxes q_i and the efficiency of heat internal sources q_{V} [W/m^3],
- the heat transfer coefficients (function) h_j = const or $h_j = h_j(\mathbf{x})$ [W/m^{2}K],
- thermophysical parameters: the heat conduction coefficient (function) $k_k = k_k(T)$ [W/m K], the specific heat $c_k = c_k(T)$ [J/kg K], the density of the material(s) $\rho_k = \rho_k(T)$

[kg/m^3], or the heat capacity $C_k = c_k\rho_k$ [J/m^{3}K] whose in some specific cases may be functions of the temperature,
- the initial distribution of temperature T_0 (in most cases, it is a constant value over the whole area/space Ω),
- the geometry (dimensions) of the analysed object,

then the solution of the initial boundary problem (1) - (4) permits to find the result, *i.e.* the temperature distribution in each point of the area Ω at every moment $t\in(0,\infty)$.

Let's denote the set of given above parameters as

$$\mathbf{S} = \left\{q_1, \cdots q_{N_\mathrm{q}}, h_1, \cdots h_{N_\mathrm{h}}, k, c, \rho, q_\mathrm{V}, T_0, \text{dimesions}\right\} \tag{5}$$

In many engineering problems analysed in practice some of the physical quantities in the set **S** (let's denote them as $\mathbf{S}_N \subset \mathbf{S}$) are, however, unknown, but it is possible to measure the temperature at selected points on the surface or inside the investigated object in the course of the process. The question arises whether it is possible to determine the elements of the set $\mathbf{S}_N$ making use of equations (1) - (4) and basing on the known temperature at the measuring points? The problem quoted in such a way has been called *Inverse Heat Transfer Problem* (*IHTP*).

The question quoted above cannot be answered unequivocally, due to errors committed while measuring the temperature and the erroneous determination of the parameters treated as present values (the set $\mathbf{S}\backslash\mathbf{S}_N$). From the mathematical point of view this problem is called an *ill-posed* or *ill-conditioned* problem (*cf.* [1], [4], [9], [10]). Looking a bit ahead we may assume that the final set of equations concerning this problem takes the form

$$\mathbf{Ay} = \mathbf{b} \tag{6}$$

where the operator **A** and the vectors **y** and **b** belong to the respective mathematical spaces are not determined here; the vector **y** consists of the elements of the set $\mathbf{S}_N$, whereas the vector **b** results from the measurements of temperature. If the problem is *well-posed*, this means that (Hadamard's conditions [9], [11]):
- for each vector **b** there is an unique solution of the set of equations (6),
- the solution is stable due to errors in the determination of the vector **b**, in other words when the error in the determination of the temperature aims at zero (and thus the error of the vector **b**), then the error in the solution (vector $\mathbf{y} = \mathbf{A}^{-1}\,\mathbf{b}$) aims at zero, too.

Most inverse problems do not satisfy these conditions, but that does not mean that in many cases it would not be possible to reach solutions which are useful from the engineering point view (*e.g.* [1], [5], [6], [9]).

2. VARIOUS KINDS OF INVERSE PROBLEMS

There are various kinds of inverse problems concerning the heat transfer, depending on the composition of the set $\mathbf{S}_N$, *viz.*
- *Inverse boundary problems* when $\mathbf{S}_N = \left\{q_1, ..., g_{N_\mathrm{q}}, h_1, ..., h_{N_\mathrm{h}}\right\}$, *i.e.* when the boundary conditions are not known.
- *Inverse parameter or function estimation problems*, when the solution aim is the determination of one or more thermophysical parameters characterising the properties of the material (or materials) constituting the investigated object ($\mathbf{S}_N = \left\{k_1, ..., k_{N_\mathrm{k}}, c_1, ..., c_{N_\mathrm{k}}, \rho_1, ..., \rho_{N_\mathrm{k}}\right\}$).

- *Inverse geometrical problems*, when we want to determine one or more quantities describing the geometry of the analysed area, *i.e.* when, for instance, $\mathbf{S}_N = \{\text{dimension of geometry, } e.g.\ R, H\}$.
- *Inverse initial problems*, when initial temperature is wanted - $\mathbf{S}_N = \{T_0\}$.
- *Inverse mixed problems*:

$$\mathbf{S}_N = \left\{q_1,...,g_{N_q}, h_1,...,h_{N_h}, k_1,...,k_{N_k}, c_1,...,c_{N_k}, \rho_1,...,\rho_{N_k}, T_0, R, H\right\}.$$

Literature quotes numerous techniques (methods) of formulating and solving inverse problems (*cf.*, *e.g.* the list of techniques quoted by Özisik [7],[11]), in most cases, however, they are reduced merely to the determination of the minimum of the adequately defined object function, applying various methods of minimization by various ways of stabilizing the solution. In this paper only the fundamental, most often applied method will be dealt with.

3. Formulation of the Inverse Problem

Let's assume that

- the area of the investigated object amounts to Ω, being not necessarily homogenous, it may, for instance consist of several areas with different mechanical and thermal properties, is bounded by N_b surfaces,
- the distribution of temperature in the area Ω can be described by the equations (1) - (4),
- it is possible to measure the temperature on the boundary of the area at N_p points with known coordinates and at N_t moments (point of time) - N_t measurements,
- N_n elements (parameters) in the set **S** (equation (5)) are unknown. Let's order these parameters, called *design parameters* (DP), in the form of a vertical column (vector) $\mathbf{Y}^\mathbf{T}$ where the symbol $\{.\}^\mathrm{T}$ denotes the transpose of the vector/matrix.

We are trying to find such a vector **Y** where the square of temperature differences measured at the measuring points **U** and the respective temperatures **T**(**Y**) obtained by solving the problem (1) - (4) is the smallest, *i.e.* we are trying to find **Y** from the condition

$$\Delta(\mathbf{Y}) = \left[\mathbf{T}(\mathbf{Y}) - \mathbf{U}\right]^\mathrm{T}\left[\mathbf{T}(\mathbf{Y}) - \mathbf{U}\right] \xrightarrow{\mathbf{Y}} \min \tag{7}$$

The columns **T** and **U** contain the respective temperatures determined by solving a direct problem and those measured at N_p measuring points in N_t moments of time, *viz.*

$$\mathbf{T}^\mathrm{T} = \left\{T_1^{(1)},\ldots,T_{N_p}^{(1)},\ldots,\quad T_1^{(i)},\ldots,T_{N_p}^{(i)},\ldots,\quad T_1^{(N_t)},\ldots,T_{N_p}^{(N_t)}\right\}^\mathrm{T} \tag{8}$$

$$\mathbf{U}^\mathrm{T} = \left\{U_1^{(1)},\ldots,U_{N_p}^{(1)},\ldots,\quad U_1^{(i)},\ldots,U_{N_p}^{(i)},\ldots,\quad U_1^{(N_t)},\ldots,U_{N_p}^{(N_t)}\right\}^\mathrm{T} \tag{9}$$

In both these formulae the subscript denotes the number of measuring point and the superscript - the number of the given measurement.
The functional (7) is an *objective function* and if developed, it takes the following form

$$\Delta\left(Y_1,\ldots Y_{N_n}\right) = \sum_{k=1}^{N_t}\left(\sum_{i=1}^{N_p}\left(T_i^{(k)}(\mathbf{Y}) - U_i^{(k)}\right)^2\right) \xrightarrow{\mathbf{Y}} \min \tag{10}$$

The necessary minimum condition of this functional is reduced to N_n equations expressed as

$$\frac{\partial\Delta(\mathbf{Y})}{\partial Y_j} = 2\frac{\partial\mathbf{T}^\mathrm{T}(\mathbf{Y})}{\partial Y_j}\left[\mathbf{T}(\mathbf{Y})-\mathbf{U}\right] = 0 \;\text{ dla }\; j=1,\ldots,N_\mathrm{n} \tag{11}$$

or in the form of a matrix

$$\mathbf{Z}^\mathrm{T}(\mathbf{Y})\left[\mathbf{T}(\mathbf{Y})-\mathbf{U}\right] = \begin{bmatrix} \frac{\partial T_1^{(1)}}{\partial Y_1} & \frac{\partial T_2^{(1)}}{\partial Y_1} & \cdots & \frac{\partial T_{N_\mathrm{p}}^{(N_\mathrm{t})}}{\partial Y_1} \\ \frac{\partial T_1^{(1)}}{\partial Y_2} & \frac{\partial T_2^{(1)}}{\partial Y_2} & \cdots & \frac{\partial T_{N_\mathrm{p}}^{(N_\mathrm{t})}}{\partial Y_2} \\ \vdots & \vdots & & \vdots \\ \frac{\partial T_1^{(1)}}{\partial Y_{N_\mathrm{n}}} & \frac{\partial T_2^{(1)}}{\partial Y_{N_\mathrm{n}}} & \cdots & \frac{\partial T_{N_\mathrm{p}}^{(N_\mathrm{t})}}{\partial Y_{N_\mathrm{n}}} \end{bmatrix}_{\{N_\mathrm{n}\times N_\mathrm{p}\cdot N_\mathrm{t}\}} \begin{Bmatrix} T_1^{(1)}-U_1^{(1)} \\ T_2^{(1)}-U_2^{(1)} \\ \vdots \\ T_{N_\mathrm{p}}^{(N_\mathrm{t})}-U_{N_\mathrm{p}}^{(N_\mathrm{t})} \end{Bmatrix}_{\{N_\mathrm{p}\cdot N_\mathrm{t}\times 1\}} = \mathbf{0} \tag{12}$$

The matrix $\mathbf{Z}$ with the dimensions $N_\mathrm{n}\times N_\mathrm{p}\cdot N_\mathrm{t}$ is called the *sensitivity matrix*; its elements are the respective derivatives

$$Z_{ij}^{(k)} = \frac{\partial T_i^{(k)}}{\partial Y_j} \tag{13}$$

called *sensitivity coefficients* (SC). They determine how far the j-th design parameter Y_j affects the temperature in the i-th measurement point at the k-th moment (measurement). The low value of this index means that a large change of the parameter Y_j leads to a small – sometimes even immeasurable – change of temperature in the measuring point. Therefore, the results of such measurements may prove to be of no use from the viewpoint of inverse analysis, whose aim it is to determine (estimate) this parameter.

4. Algorithm for the Solution of Inverse Problem

Further on we shall discuss an iterative way of finding the minimum of the functional (7), and thus also an iterative solution of the inverse problem. The presented algorithm is called *sensitivity coefficients method* (*cf.* [9], [12], [14]).

Let’s assume that the temperature in the n-th iterative step can be replaced by its Taylor expansion versus the temperature known from the iteration (n - 1), obtained by the solving the direct problem concerning the previously determined design variables $\mathbf{Y}^{(n-1)}$. Thus, reducing the expansion to a linear term, the temperature may be expressed by the relation

$$\mathbf{T}\left(\mathbf{Y}^{(n)}\right) = \mathbf{T}\left(\mathbf{Y}^{(n-1)}\right) + \mathbf{Z}\left(\mathbf{Y}^{(n-1)}\right)\left(\mathbf{Y}^{(n)}-\mathbf{Y}^{(n-1)}\right) \tag{14a}$$

or in a shortened way as

$$\mathbf{T} = \mathbf{T}^{(n-1)} + \mathbf{Z}\left(\mathbf{Y}-\mathbf{Y}^{(n-1)}\right) \tag{14b}$$

where the following expressions are introduced: $\mathbf{Y}^{(n)} = \mathbf{Y}, \quad \mathbf{T}\left(\mathbf{Y}^{(n)}\right) = \mathbf{T}$.

Substituting the relation (14) into the condition (12) we get the following algebraic set of equations expressed in matrix form

$$\mathbf{Ay} = \mathbf{b} \tag{15}$$

where

$$\mathbf{A} = \mathbf{Z}^{\mathrm{T}}\mathbf{Z},\ \mathbf{y} = \mathbf{Y}^{(n)},\ \mathbf{b} = \mathbf{Z}^{T}\left(\mathbf{T}\left(\mathbf{Y}^{(n-1)}\right) - \mathbf{U} + \mathbf{Z}\mathbf{Y}^{(n-1)}\right). \tag{16}$$

<u>The successive stages of the algorithm are as follow:</u>

<u>Stage I</u>: *Initial guess for* **Y**, $n = 0$, i.e. $\mathbf{Y}^{(0)} = \mathbf{Y}^{*}$

These values may, in principle, be arbitrary, but ought to approximate as much as possible the expected solution (which, of course, requires some experience and knowledge of the analyzed problem). In result we get the temperature at the measurement point $\mathbf{T}\left(\mathbf{Y}^{(0)}\right) = \mathbf{T}\left(\mathbf{Y}^{*}\right) = \mathbf{T}^{*}$

<u>Stage II</u>: *Determination of the sensitivity coefficient matrix* **Z**, and next, from equation (15), the new values of the design variables $\mathbf{Y}^{(n)}$ and by means of the relation (14) the corresponding temperature **T**.

<u>Stage III</u>: *Verification of the convergence condition of the solution*, *e.g.* expressed like this (*cf.* [4], [11])

$$\left\|\mathbf{Y}^{(n)} - \mathbf{Y}^{(n-1)}\right\| < \varepsilon \tag{17}$$

where ε denotes the present admissible difference between two successive solutions. If the condition (17) is not satisfied, we assume that $\mathbf{Y}^{*} = \mathbf{Y}^{(n)}$ and n becomes n - 1 and then the stage II of the algorithm is repeated (perform next iteration).

5. SENSITIVITY ANALYSIS

When preparing an experiment in which temperature is the basic quantity for further analyses and the equations which describes its distribution in the analyzed object is a heat conductivity equation (1) including the respective boundary conditions, then we must assess to which extent the respective quantities in the set **S** influence the temperature measured at the selected measuring points. The sensitivity analysis being a part o the experiment applied in inverse problems permits to determine

- the best position of the measuring points,
- which ones from among the unknown quantities can be determined basing on temperatures measured in selected boundary points or inside the object,
- those quantities (*e.g.* geometrical dimensions, ambient temperature), which must be measured (or determined experimentally) as accurately as possible, being indispensable for the solution of the heat transfer problem, and which of them are less important and may be only assessed,
- those quantities which due to the accuracy of the instruments measuring the temperature cannot be used as design variables.

Measures of the influence on temperature are the previously defined sensitivity coefficients (13), which cannot, however, be compared in their original form. Therefore, in order to assess the influence of various parameters we apply dimensionless sensitivity coefficients.

The *dimensionless sensitivity coefficient* of the parameter Y_j is defined as (*cf.* [11])

$$Z_{Y\,j} = \frac{\partial T}{\partial Y} \frac{Y_j}{\left(qL/k\right)} \tag{18}$$

where q is the heat flux which essentially affects the temperature distribution in the area Ω; L is the characteristic dimension of the investigated sample (*e.g.* its height), and k is the heat conductivity coefficient of the material of which the sample has been made.

6. EXEMPLARY PROBLEM

In order to illustrate conceptions mentioned above and the methods of such a procedure let's formulate an exemplary technical problem (*cf.* [2], [12], [14], [15], [16],).

Let's assume that an axially symmetrical temperature field is analyzed in an area consisting of two cylinders – $\Omega_m = \Omega_m\left(r,\varphi,z\right) = \Omega_m\left(\mathbf{x}\right)$ $m = 1,\ 2$ (see Fig. 1) with different physical properties: the thermal conductivity $k_m = k_m\left(T\right)$, the specific heat $c_m = c_m\left(T\right)$ and the density ρ_m Let's also assume that the parameters of the first cylinder depend on temperature. Let the base of the first cylinder be heated by the heat source q_s. Finally, we assume the third kind of the boundary conditions on the lateral surfaces.

Mathematically the distribution of temperature on the particular subareas is expressed by Fourier's equation noted in the axial symmetric $\left(\mathbf{x} = \left(r,\varphi,z\right) = \left(r,z\right)\right)$ cylindrical co-ordinate system

$$\nabla_r\left[k_1\left(T\right)\nabla T_1\left(\mathbf{x},t\right)\right] = \rho_1\, c_1\left(T\right)\frac{\partial T_1\left(\mathbf{x},t\right)}{\partial t} \quad \text{for } \mathbf{x}\in\Omega_1 \text{ and } t\in\left(0,\infty\right) \tag{19}$$

$$\nabla_r\left[k\ \nabla T_2\left(\mathbf{x},t\right)\right] = \rho_2\, c_2\frac{\partial T_2\left(\mathbf{x},t\right)}{\partial t} \quad \text{for } \mathbf{x}\in\Omega_2 \text{ and } t\in\left(0,\infty\right) \tag{20}$$

supplemented by the respective boundary conditions

$$\begin{aligned}
-k_1\left(T\right)\frac{\partial T}{\partial z} &= q_s, && r\le R \quad \text{and} \quad z=0\\
-k_1\left(T\right)\frac{\partial T}{\partial r} &= h_1\left(z\right)\left(T_1 - T_a\right), && r=R \quad \text{and} \quad 0\le z\le H_1\\
-k_2\frac{\partial T}{\partial r} &= h_2\left(z\right)\left(T_2 - T_a\right), && r=R \quad \text{and} \quad H_1\le z\le H_2\\
-k_2\frac{\partial T}{\partial r} &= h_t\left(z\right)\left(T_2 - T_a\right), && r\le R \quad \text{and} \quad z=H_2
\end{aligned} \tag{21}$$

the conditions of continuity (without resistance h_q) on the boundary of the subareas $r\le R$ i $z = H_1$

$$\begin{aligned}
T_1 &= T_2\\
k_1\left(T\right)\frac{\partial T_1}{\partial z} &= k_2\left(T\right)\frac{\partial T_2}{\partial z}
\end{aligned} \tag{22}$$

and the initial conditions

$$T_m(\mathbf{x},0) = T_m^0(\mathbf{x}),\ m = 1,\ 2 \tag{23}$$

The symbols h_1, h_2, h_t denote in these formulae the respective hear transfer coefficients on the lateral surfaces of the cylinders and on the base of the second cylinder, whereas T_a denotes the ambient temperature.

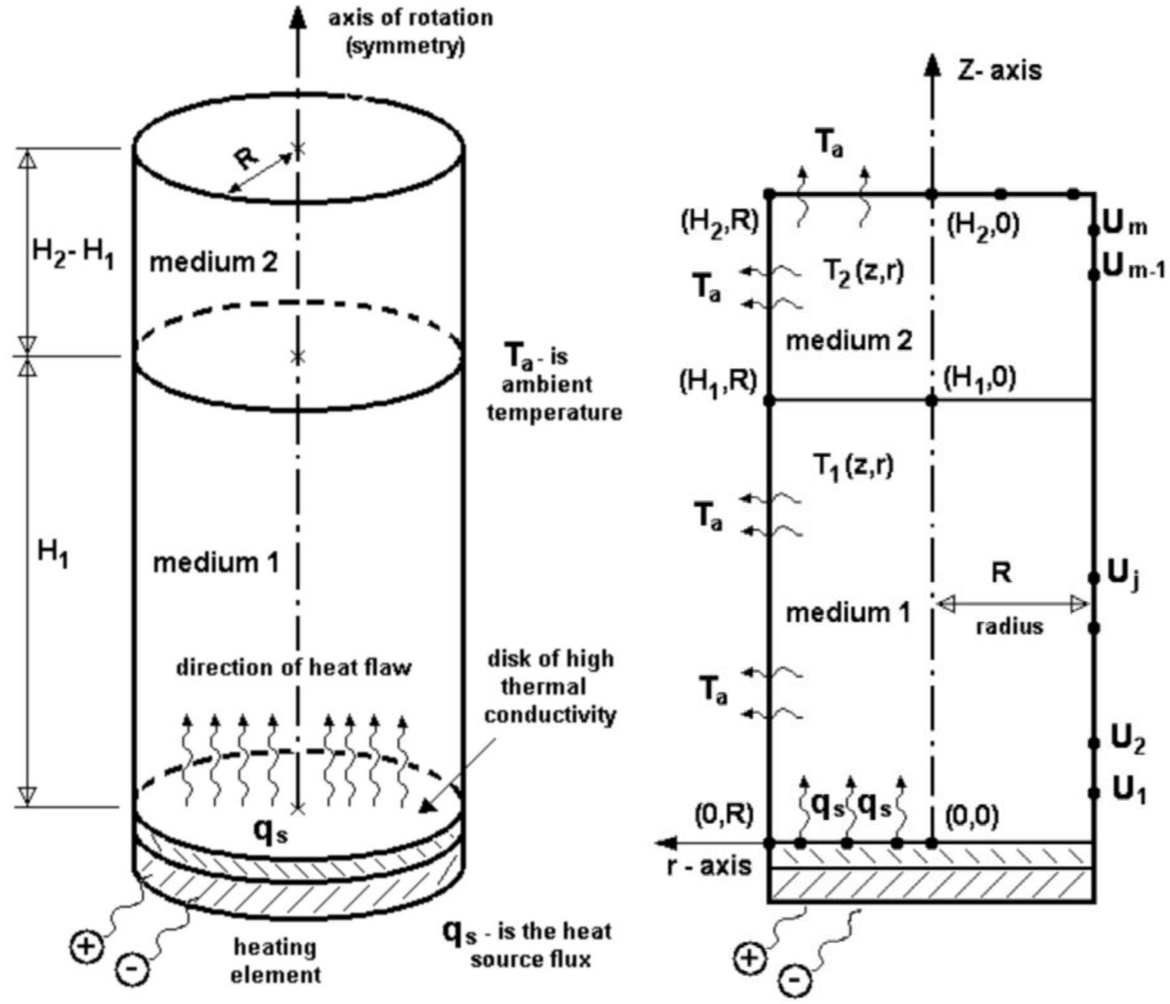


Figure 1. The analyzed area

Let's, moreover, assume that the temperature is measured on the lateral surface of the sample at *m* equally spaced points, making use of a infrared camera, so that in our further analysis we shall have at our disposal a large number of extremely accurate measurement data. We also assume that the upper cylinder has been made of homogeneous material (called *reference material*) with known physical parameters, determined with a high accuracy.

The temperature field depends on the following parameters or functions:

- material parameters: $k_1(T), c_1(T), \rho_1, k_2, c_2, \rho_1, \rho_2$
- boundary parameters: $q_s, h_1(z), h_2(z), h_t, T_a, T_0$
- measurement parameters: position of the initial and final point of the read-out of the temperature – z_{pocz}, z_{kon}
- geometrical parameters: R, H_1, H_2

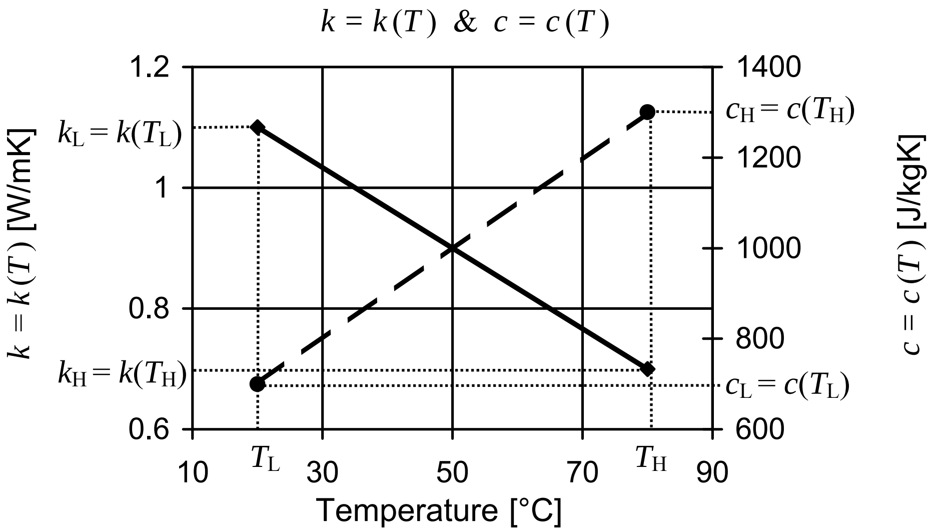


Figure 2. The temperature dependence of the thermal conductivity and the specific heat

Let's assume that the material functions change linearly form the values $k_L = k(T_L)$ and $c_L = c(T_L)$ - at lower temperature T_L – to the values $k_H = k(T_H)$ and $c_H = c(T_H)$ – at the temperature T_H (see Fig. 2). The heat transfer coefficient changes linearly versus the height (variable *z*) from the value $h_B = h(0)$ at the base to $h_M = h(H_1)$ at the boundary of the cylinders and to $h_T = h(H_2)$ at the highest point of the side surface (see Fig. 3). Let the heat transfer coefficient on the upper surface be $h_t = \text{const}$.

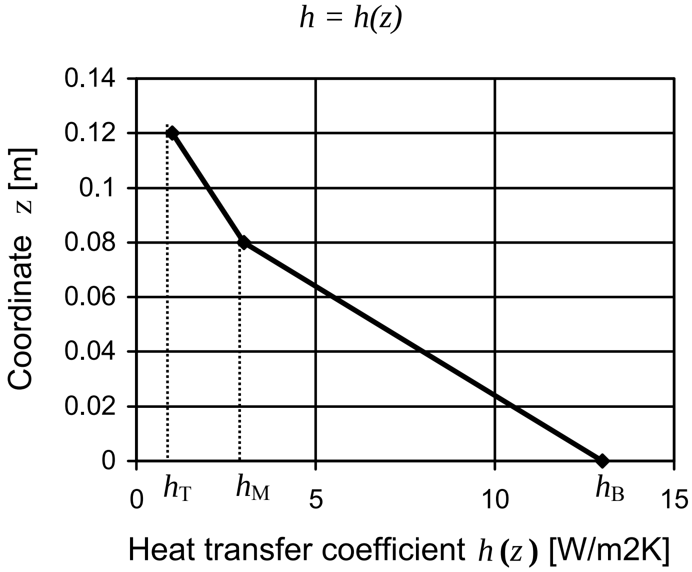


Figure 3. Changes of the heat transfer coefficient along the side surfaces of both cylinders

7. Dimensionless Sensitivity Indices

In the considered problem the sensitivity coefficients for the respective measurement points were determined by approximating the derivatives by the respective difference quotients, *i.e.*

$$z_{Y_j} = \frac{\partial T}{\partial Y_j} \frac{Y_j}{\left(q_s L / k\right)} \approx \frac{T\left(Y_j + \Delta Y_j\right) - T\left(Y_j\right)}{\Delta Y_j} \frac{Y_j}{\left(q_s L / k\right)}, \; j = 1, \dots, 18 \tag{24}$$

in which case the vector **Y** provides 18 quantities:

$$\mathbf{Y} = \left\{ k_L, k_H, c_L, c_H, \rho_1, k_2, c_2, \rho_2, q_s, h_B, h_M, h_T, h_t, z_{pocz,} z_{kon}, R, H_1, H_2 \right\}$$

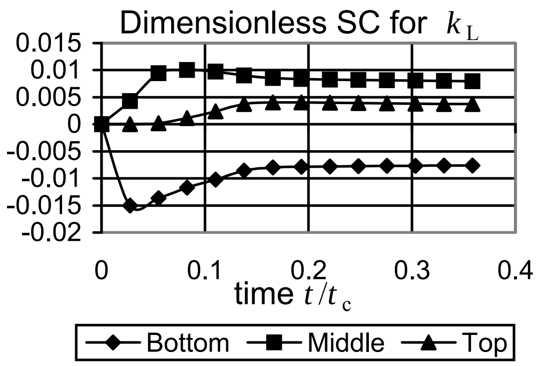
Thus, in order to determine the coefficients 1 + 18 the direct problems (1) - (4) must be solved: first for those concerning the values of all the parameters, and then successively for the changed value of only one of them increased by ΔY_j (where index *j* is changing from 1 to 18). From each solution, the temperatures in equation (24), in the respective measuring points are used for all the present moments of time. The eight diagrams on Fig. 4 and 5 illustrate only some results of the analysis of the considered problem. Each diagram contains three curves showing how the value of the dimensionless coefficients of sensitivity changes in time at three measurement points on the side surface of the cylinders: the lowest one h_B (at the base), that in the middle h_M and that in the highest position h_T. In all these diagrams the axis of abscissas is the axis of dimensionless time defined as

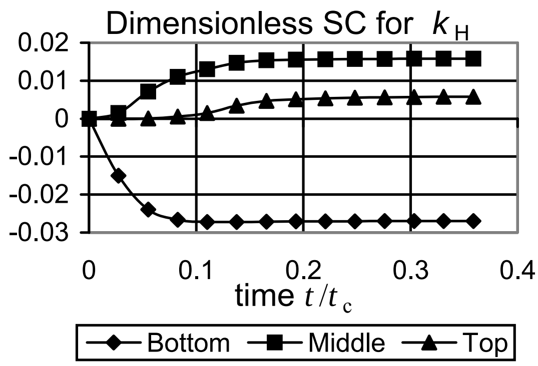
$$\tau = \frac{k_L t}{c_L \rho_1 H_2}$$

The subsequent diagrams present the dimensionless sensitivity coefficients concerning the following parameters: $k_L, k_H, c_L, c_H, h_B, h_M, h_T, h_t$

The analysis of these diagrams (and also of those not presented in this paper) and of the solution of many inverse problems (concerning the configuration described above) leads to the following conclusions:

- The optimal dimensionless time of heating is contained within the range (interval) $\tau_k \in \left(0.15;\ 0.25\right)$ because the effect of most parameters on the temperature distribution becomes stabilized (frequently on a level approaching zero) or changes at the same rate (the curves of different parameters are almost parallel).
- In the most measurement points the temperature is most affected by such parameters as the heat flux q_s, the ambient temperature T_a, the heat transfer coefficient h_B in the vicinity of the heating element, the thermal conductivities k_L, k_H and the specific heat c_L. The second parameter describing specific heat - c_H - exerts a considerable influence on the temperature only at the measurements points situated at the half height of the sample, so that its determination is connected with larger errors than other design parameters.





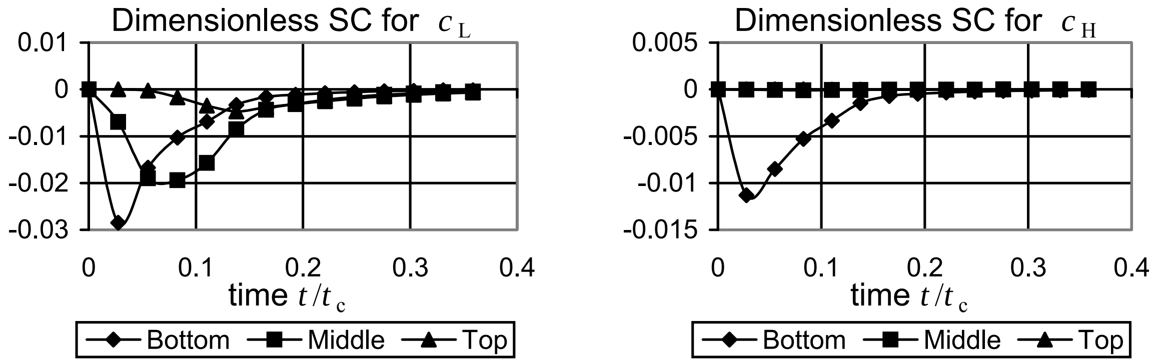
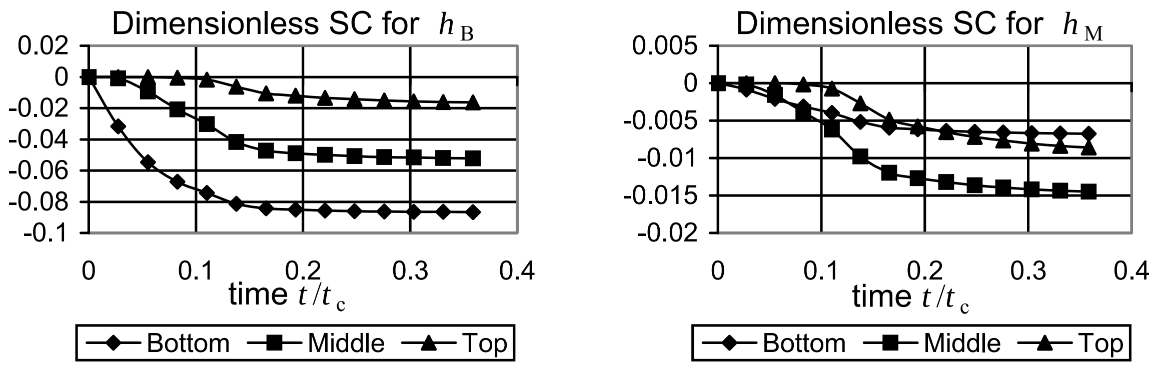


Figure 4. Dimensionless sensitivity coefficients for thermophysical parameters

- The heat transfer coefficients h_t, h_T on the surfaces remote from the heating element (side surface of the upper cylinder and the upper base) influences the temperature measurements least.
- Quantities with large sensitivity coefficients are found in the inverse analysis rather accurately, even in the case of considerable (about 1 K) measurement errors (assuming that measurements are taken by means of highly sensitive infrared camera [3], [8]).
- Geometrical quantities indicate a considerable influence on the temperature distribution in the object, but most often solving inverse problems does not estimate them, because they can be measured with a good accuracy.
- The influence of the parameters k_2, c_2, ρ_2 (reference material) is only small, which is advantageous because of possible errors occurring in the case of applying classical methods of measurements – it is assumed they are known as the reference parameters for inverse analysis.
- From the point of view of IR measurements an extraordinarily sensitive parameter is the co-ordinate of the point of cylinder side surface corresponding to the first point of the recording path of the temperature. As the most measurements taken in the upper part of cylinder are less sensitive, the sensitivity coefficient for the last upper measurement point is smaller than SC of the lower point.
- The smaller sensitivity for given parameters, the less accurate is its estimation (see Fig. 10).
- As might be expected basing on the analysis of the heat transfer equation (1), the sensitivity coefficients concerning the specific heat and mass density of the tested material are identical.



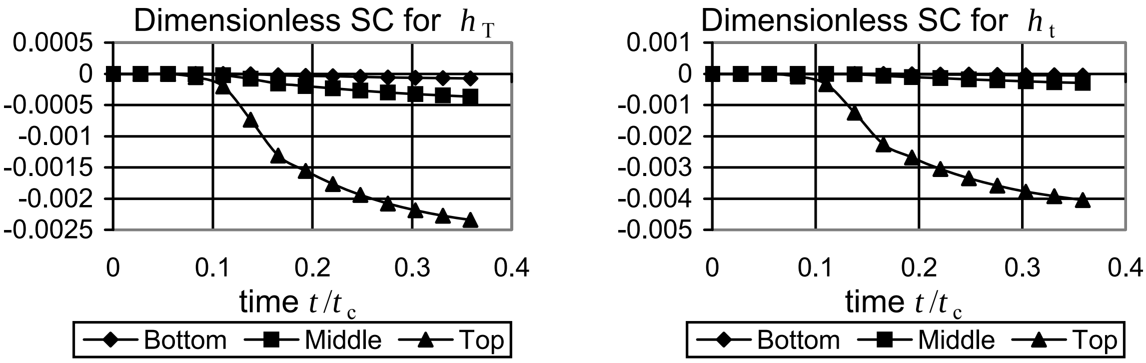


Figure 5. Dimensionless sensitivity coefficients for the boundary conditions

8. Factors Deciding About the Accuracy of the Solution of IHTP

In the first chapter of this paper it has been mentioned that inverse problems are, on the whole, problems, which have been ill-posed due to the errors occurring in the measurements of temperature. A decisive role in the process of solving such problems is played by the sensitivity matrix, from which the final matrix of the set of equations is generated (15), *i.e.*

$$\mathbf{A} = \mathbf{Z}^T \mathbf{Z}, \tag{25}$$

Small values of the SC lead to a low determinant of the system

$$\det[\mathbf{A}] = \det\left[\mathbf{Z}^T \mathbf{Z}\right] \approx 0 \tag{26}$$

and this means that the system is wrongly conditioned. This determinant can also be small (approaching or equal to zero), if the SC of two design parameters are linearly dependent. Hence the conclusion that before a concrete inverse problem is solved, the sensitivity of the experiment must be carefully analyzed and the influence of various factors on the value of the determinant $\det[\mathbf{A}]$ of the matrix of the given system ought to be investigated.

The factors which decide about the value of the determinant of the matrix **A** are among others,

1. the total time of heating t_k (when the number of measurements being constant),
2. the number of measurements in the total time of heating t_k ,
3. the choice and number of measurement points,
4. the way of determining and the values of the SC,
5. the values of some parameters (with high sensitivity coefficients) which strongly affecting the temperature values at the measurements points,
6. the position of measurement points.

Figures 6 & 7 illustrate the influence of the factors 1 - 3 in the case of the exemplary problem dealt with in the sixth chapter.

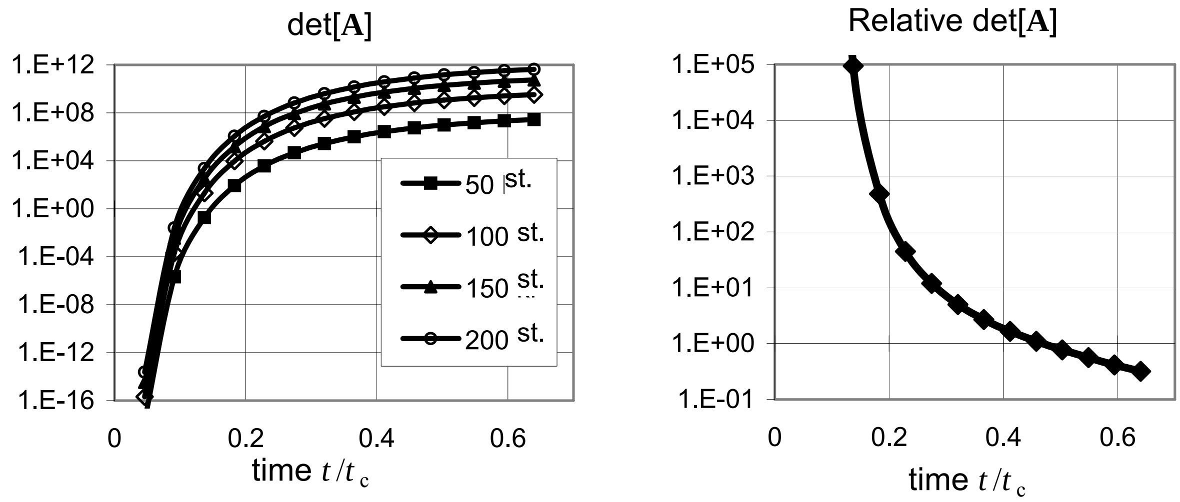


Figure 6. The determinant value changing with increasing time of heating and the different number of measurement point

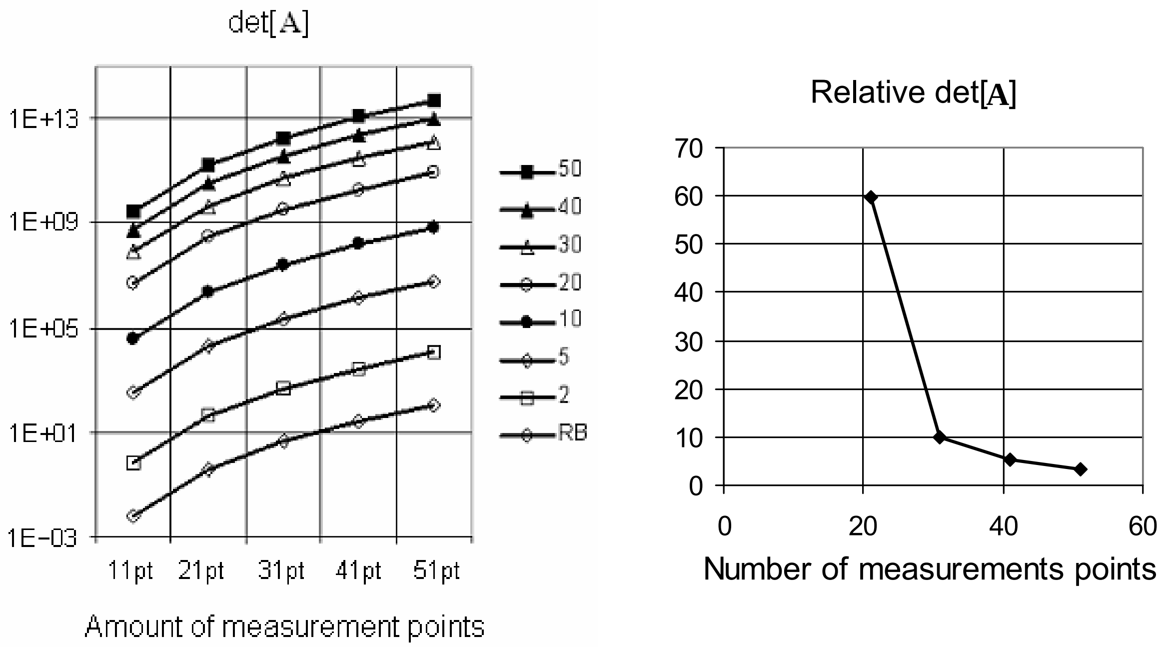


Figure 7. The determinant relative value changing at a constant time of heating and changing number of measurement points

The diagram on the left-hand side of Fig. 6 shows how the value of the determinant changes in the case of the problem with a constant number of time steps (cases 50, 100, 150 and 200 steps) if the dimensionless total time of heating τ_k rises. The relative increase of the value of the determinant is the same, independently of the number of measurement points – diagram on the right-hand side of Fig. 6. As we can see, in the case of the dimensionless time of heating, amounting to about 0.2, we may assume that the increase of determinant $\det[\mathbf{A}]$ is no longer of importance –

which confirm earlier conclusions derived from the sensitivity analysis. Analogically, Fig. 7 illustrates the changes of the value of the $\det[\mathbf{A}]$ when the number of measurement points differs, the time of heating being constant (in the legend to this diagram the multiplicity of the τ_k for a basic solution RB has been given). The increase of the determinant value does not depend on the total time τ_k - diagram on the right-hand side of Fig. 7.

9. Exemplary Solution of an Inverse Problem

Here we are going to present a way of solving an exemplary inverse problem, the area, equations and boundary conditions (BC) of which have been dealt with in sixth chapter.

Let's assume that the results of temperature measurements with an IR camera on the side surfaces of the cylinders as in Fig. 1. are used to determine the thermal conductivity and specific heat of the cylinder at the bottom.

Besides that, we will assume that in an actual sample both quantities $k_1 = k_1(T)$ and $c_1 = c_1(T)$ are functions of temperature (power functions of the third order), and the heat transfer coefficient on the entire side surface of the sample $h = h(z)$ $\left(\text{where } z \in (0, H_2)\right)$ changes like a square function.

Stages of the problem solution

Stage I: Simulated measurement of temperatures

In order to simulate the measurement of temperature the direct problem (1) - (4) is solved with all the parameters taken from the set **S**. The co-ordinates of the measurement points result from the assumed number of subintervals, into which the side surface of the sample has been divided (Fig. 1). The random errors with a given maximum value (*e.g.* +/-1°) are added to calculated temperatures.

Stage II: The choice of design parameters

The aim of the simulated experiment is to determine the two functions mentioned above - $k_1 = k_1(T)$ and $c_1 = c_1(T)$, and also other design parameters concerning the temperature distribution which are unknown or which can be assessed with a large error. We know, for instance, the amount of electric energy consumed by the heater, but not its efficiency. Another problem is the determination of the heat flux given up to the environment by the lateral and upper surface of the sample. Wanting to describe the conditions of heat exchange with environment we may assume the second and third kind boundary conditions, although the parameters, which express these conditions, are rather difficult to be assessed, even the case of extensive engineering experience, particularly when the surface is small as in the case of the samples presented in Fig. 1. In the case of the II kind of BC we may have to do with a large number of design parameters, because the boundary of the area should be divided into numerous sub-areas and for each of then the heat flux must be determined.

In this problem, three functions: $k_1 = k_1(T), c_1 = c_1(T), h = h(z)$ must be determined. We assume that we search their linear or partly linear (broken line) approximations. Simulating the measurements they are assumed to be power functions. Therefore the vector of design parameters (DP) consists of the following nine elements

$$\mathbf{Y} = \{k_\mathrm{L}, k_\mathrm{H}, c_\mathrm{L}, c_\mathrm{H}, q_\mathrm{s}, h_\mathrm{B}, h_\mathrm{M}, h_\mathrm{T}, h_\mathrm{t}\}\,. \tag{27}$$

As known values resulting from measurements the following quantities were used: $\rho_1, k_2, c_2, \rho_2, z_{\mathrm{pocz},} z_{\mathrm{kon}}, R, H_1, H_2$.

Stage III: Assumption of the initial values of the design parameters

Knowing the values of DP, assumed in the simulation of measurements (Stage I), various initial values of DPs are applied. In the most cases they differed from accurate values by 50-150 %. From numerous varying problems it results that it is of more advantage to assume lower starting values of the DPs than the expected solution, because in the case of thermal quantities the solution obtained in the first iteration is negative, so that the majority of solving programs must be suspended. The initial values of DPs are recorded in the vector $\mathbf{Y}^*$.

Stage IV: Iterative solution of the problem

The temperatures at the measurement points, i.e. the vector $\mathbf{T}^*$, are found by solving the direct problem with assumed parameters $\mathbf{Y}^*$. Next the matrix $\mathbf{Z}$ of the sensitivity coefficients and equation set matrix $\mathbf{A}$ (14) and free terms vector $\mathbf{b}$ are generated. Results of the solution of this set are in a new vector $\mathbf{Y}$.

Stage V: Checking the conditions of convergence of the solution

If the new vector Y do not satisfy the condition

$$\left\|\mathbf{Y}-\mathbf{Y}^*\right\| < \varepsilon \tag{28}$$

we may assume that $\mathbf{Y}^* = \mathbf{Y}$ and repeat the calculations starting with stage II.

Calculations are carried out making use of the ANSYS – a finite element method package, and the authors' own procedures utilizing ANSYS Parametric Design Language (APDL). Fig. 8. provides the values of several DPs in successive iterations, related to exact data (though from nonlinear simulation). The next two diagrams in Fig. 9 & 10 present the results of solutions concerning the material functions $k_1 = k_1(T)$ and $c_1 = c_1(T)$ for various total dimensionless times of heating (cases: $\tau_k = 0.275, 0.23, 0.18, 0.14, 0.9$) and curves assumed in the simulation of measurements. Due to the thermal conductivity, the best solution is that one when $\tau_k = 0.23$, but for the specific heat when $\tau_k = 0.275$, It confirms earlier conclusions resulting from the analysis of sensitivity. In the case of the shorter time, the obtained solutions are encumbered with considerable errors, particularly if $\tau_k = 0.14$ or 0.9. In comparison with other thermophysical parameters the value of the c_{H} is estimated with a greater error - what also confirms the conclusions that have been put earlier.

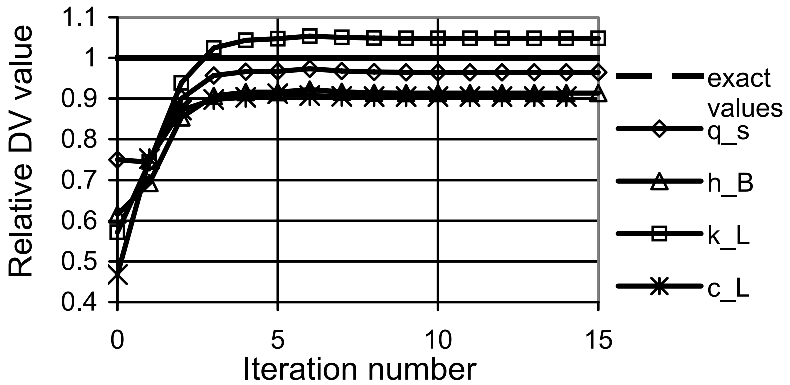


Figure 8. Design parameters in successive iterations

For the heat transfer coefficients, more accurately is evaluating process of the $h_{\mathrm{B}}, h_{\mathrm{M}}$ that describe the boundary conditions near the bottom base of the sample. The remaining quantities, which have relatively small SCs (like h_t, h_T), are estimated with considerable errors and influence – to a large extent – the obtained values of the other DPs. An assessment of these two DPs – based on the knowledge of the physical aspect of the experiment – seems to be a better approach.

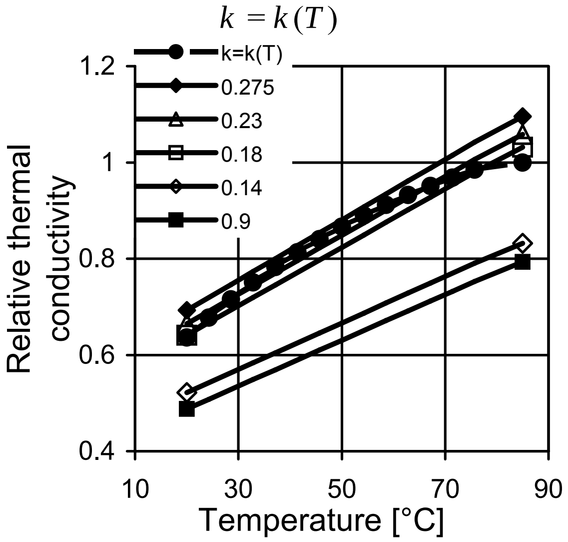


Figure 9. Thermal conductivity at various times of heating

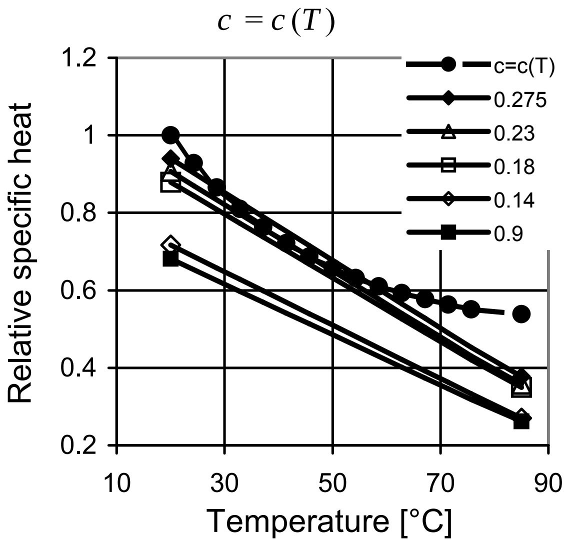


Figure 10. Specific heat determined at various times of heating

10. Final Remarks

The remaining problems have been dealt with only in short way; those interested in more details are advised to refer to relevant literature.

- *Determination of the sensitivity coefficients.* Dealing with the problem discussed above, a differential approximation of the derivative is applied to determine these coefficients (*cf.* equation (24)). The increment ΔY of the design parameters amounts from 0.001 to 0.0001 of its value in the given iteration ($\Delta Y = (0.0001 \div 0.001)Y$). The values of the increment result from analysis of direct problems solutions obtained at various assumed increments. Generalizing, we may say that the values of the sensitivity coefficients do not change much if $\Delta Y = (0.0001 \div 0.001)Y$. Smaller values of ΔY lead to considerable errors resulting from computer operations. Literature (*e.g.* [9], [11]) suggests for boundary conditions of the second kind to determine the sensitivity coefficients by solving the respective initial-boundary problem resulting from the differentiation of the equation concerning this problem (heat conductivity equation) versus the given DP. Thus, the same (from the mathematical point of view) equation (1) must be solved iteratively with simplified boundary conditions: $-k\,\partial(Z_Y)/\partial n = 1$ lub 0 .
- *Other method of searching for the minimum objective function.* Measurement errors may cause oscillations or instability of the solution. One of the ways of avoiding these difficulties is the method of regularization, which has been largely discussed in [1], [9] and [13]. In this method to the classical objective function (7) a smoothing additional matrix is added multiplied by the so-called coefficient of regularization. Various methods of determining the value of this coefficient have been suggested in the mentioned

publications. A very effective method of stabilizing the solution is a method based on the iterative Levenberg-Marquardt procedure (see [11]). This and other methods have been presented comprehensibly in [1], [5], [10] and [11].

- *Stability of the solution.* This problem has been dealt with, for instance, in [1] & [4]. The authors checked the stability of the method in numerical practice, solving problems with various levels of measurement errors and various initial values of the design parameters, comparing them with the solution applied in the simulation of measurements.

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