LINEAR THERMAL EXPANSION OF THE SPHERICAL MODEL OF THE TWO-PHASE MATERIAL

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Abstract: The structure of the two-phase material can be considered as two concentric spheres. The inner sphere, the grain and the outer sphere, the cladding, have radii R_g , R_c and there material parameters are E_g , E_c (Young's modulus), μ_g , μ_c (Poisson's ratios) and α_g , α_c (coefficients of the linear thermal expansion). The expansion of such model during its heating can be solved as a thermoelasticity problem. We derived the coefficient of linear thermal expansion of the two phase material in the case when $E_g \neq E_c$, $\mu_g \neq \mu_c$ and $\alpha_g \neq \alpha_c$.

Key words: thermal expansion, two-phase material

1 Introduction

Some materials can be considered two-phase solids. For example, sintered quartz electrical porcelain contains quartz grains in glassy matrix [1]. We can visualize this structure as consisting of two concentric spheres. The inner sphere, the grain, has a radius R_g , and its material parameters are E_g (Young's modulus), μ_g (Poisson's ratio) and α_g (coefficient of the linear thermal expansion). The outer sphere, the cladding, has material constants E_c , μ_c and α_c and its radius is R_c. We can solve the expansion of this model during its heating as a thermoelasticity problem. If the temperature increase is small and inertial forces are negligible, then the temperature field and the stress field do not influence each other. We assume that grain and cladding materials are homogeneous and isotropic, which means that a radial thermal flow occurs. We also assume that the material parameters are constant in the considered temperature region.

The general formula for the coefficient of linear thermal expansion, given the above assumptions, is derived in this contribution.

2 Mathematical model

It follows from the assumptions described above, that the displacement vector $\vec{u}$ has only a radial component u_r. Deformations in a simple sphere along the spherical coordinates are [2]

$$\varepsilon_r = \frac{\partial u_r}{\partial r}, \quad \varepsilon_\vartheta = \frac{1}{r} u_r, \quad \varepsilon_\varphi = 0$$

and shear deformation is

$$\varepsilon_{r\vartheta} = \frac{1}{2r} \frac{\partial u_r}{\partial \vartheta} = 0,$$

because the radial displacement u_r does not change along the tangential direction. One can see that deformations are only in the radial and tangential directions. We can use these results for a simple sphere [2]. The radial stress in the grain is

$$\sigma_{rg}(r)=\frac{E_g c_{g1}}{1-2\mu_g}-\frac{2E_g c_{g2}}{1+\mu_g}\frac{1}{r^3}-\frac{2E_g}{1-\mu_g}\frac{1}{r^3}\int_0^r \varepsilon_{gf} r^2 dr \quad , \quad \text{for} \quad 0<r\le R_g \, , \tag{1}$$

and tangential stress is

$$\sigma_{\vartheta g}(r)=\frac{E_g c_{g1}}{1-2\mu_g}-\frac{E_g c_{g2}}{1+\mu_g}\frac{1}{r^3}-\frac{E_g \varepsilon_{gf}}{1-\mu_g}+\frac{E_g}{1-\mu_g}\frac{1}{r^3}\int_0^r \varepsilon_{gf} r^2 dr \quad , \quad \text{for } 0<r\le R_g \, , \tag{2}$$

and radial component of the displacement vector is

$$u_{rg}(r)=c_{g1}r+c_{g2}\frac{1}{r^2}+\frac{1+\mu_g}{1-\mu_g}\frac{1}{r^2}\int_0^r \varepsilon_{gf} r^2 dr \quad , \quad \text{for } 0<r\le R_g \, . \tag{3}$$

The value ε_{gf} is a free deformation during the temperature change. This deformation can be calculated using the equation

$$\varepsilon_{gf}=\int_{t_0}^{t} \alpha_g \, dt \quad ,$$

thus integral in Eqs. (1), (2), (3) can be written as

$$\int_0^r \int_{t_0}^{t} \alpha_g \; r^2 dr \; dt = \alpha_g \Delta t \frac{r^3}{3} \, ,$$

where $\Delta t = t - t_0$ is a temperature difference between initial temperature t_0 and actual temperature t.

For the cladding we have

$$\sigma_{rc}(r)=\frac{E_c c_{c1}}{1-2\mu_c}-\frac{2E_c c_{c2}}{1+\mu_c}\frac{1}{r^3}-\frac{2E_c}{1-\mu_c}\frac{1}{r^3}\int_{R_g}^r \varepsilon_{cf} r^2 dr \quad , \text{for } R_g<r\le R_c \, , \tag{4}$$

$$\sigma_{\vartheta c}(r)=\frac{E_c c_{c1}}{1-2\mu_c}-\frac{E_c c_{c2}}{1+\mu_c}\frac{1}{r^3}-\frac{E_c \varepsilon_{cf}}{1-\mu_c}+\frac{E_c}{1-\mu_c}\frac{1}{r^3}\int_{R_g}^r \varepsilon_{cf} r^2 dr \quad , \text{for } R_g<r\le R_c \, , \tag{5}$$

$$u_{rc}(r)=c_{c1}r+c_{c2}\frac{1}{r^2}+\frac{1+\mu_c}{1-\mu_c}\frac{1}{r^2}\int_{R_c}^r \varepsilon_{cf} r^2 dr \quad , \text{for } R_g<r\le R_c \, , \tag{6}$$

and the integral in Eqs. (4), (5), (6) is

$$\int_{R_g}^r \int_{t_0}^{t} \alpha_c \; r^2 dr \; dt = \alpha_c \Delta t \frac{r^3 - R_g^3}{3} \, .$$

Constants c_{g1}, c_{g2}, c_{c1}, c_{c2} can be calculated from the boundary conditions. The boundary conditions on the surface between the grain and cladding ($r = R_g$) and on the surface of the cladding ($r = R_c$) are [3]

$$u_{rg}(0)=0 \, , \quad u_{rg}(R_g)=u_{rc}(R_g) \, , \quad \sigma_{rg}(R_g)=\sigma_{rc}(R_g) \, , \quad \sigma_{rc}(R_c)=0 \, . \tag{7}$$

Substituting Eqs. (1), (3), (4) and (6) into the boundary conditions (7) we obtain four equations

$$\lim_{r\to 0}\left[c_{g1}r+c_{g2}\frac{1}{r^2}+\frac{D_g}{B_g}\frac{1}{r^2}\frac{\alpha_g \Delta t}{3}r^3 \right]=0 \, , \tag{8}$$

$$c_{g1}R_g+\frac{D_g R_g}{B_g}\frac{\alpha_g \Delta t}{3}=c_{c1}R_g+\frac{c_{c2}}{R_g^2} \, , \tag{9}$$

$$A_g c_{g1}-2D_g\frac{\alpha_g \Delta t}{3}=A_c c_{c1}-\frac{2B_c}{R_g^3}c_{c2} \, , \tag{10}$$

$$A_c c_{c1} - \frac{2B_c}{R_c^3} c_{c2} - \frac{R_c^3 - R_g^3}{R_c^3} \frac{\alpha_c \Delta t}{3} 2D_c = 0, \tag{11}$$

where we introduced abbreviated designations

$$A_g = \frac{E_g}{1 - 2\mu_g},\ A_c = \frac{E_c}{1 - 2\mu_c},\ B_g = \frac{E_g}{1 + \mu_g},\ B_c = \frac{E_c}{1 + \mu_c},\ D_g = \frac{E_g}{1 - \mu_g},\ D_c = \frac{E_c}{1 - \mu_c}.$$

By solving these equations we obtain the constants c_{g1}, c_{g2}, c_{c1}, c_{c2} :

$$c_{g1} = \frac{\alpha_g \Delta t}{3} \frac{D_g}{B_g} \left(\frac{A_g - 2B_g}{A_g - A_c} - 1 \right) - \frac{c_{c2}}{R_g^3} \left(\frac{2B_c + A_g}{A_g - A_c} - 1 \right), \tag{12}$$

$$c_{g2} = 0\ , \tag{13}$$

$$c_{c1} = \frac{\alpha_g \Delta t}{3} \frac{D_g}{B_g} \left(\frac{A_g - 2B_g}{A_g - A_c} \right) - \frac{c_{c2}}{R_g^3} \left(\frac{2B_c + A_g}{A_g - A_c} \right), \tag{14}$$

$$c_{c2} = \frac{\alpha_g \Delta t}{3} \frac{\frac{A_c}{2B_c} \frac{D_g}{B_g} \left(\frac{A_g - 2B_g}{A_g - A_c} \right) R_c^{\ 3} - \frac{D_c}{B_c} \left(R_c^{\ 3} - R_g^{\ 3} \right)}{1 + \frac{A_c}{2B_c} \left(\frac{2B_c + A_g}{A_g - A_c} \right) \frac{R_c^{\ 3}}{R_g^{\ 3}}}. \tag{15}$$

After mathematical modifications and introduction of following quantities

$$F = \frac{A_g - 2B_g}{A_g - A_c},\ G = \frac{2B_c + A_g}{A_g - A_c},\ H = \frac{A_c}{2B_c},\ J_g = \frac{D_g}{B_g},\ J_c = \frac{D_c}{B_c},\ v = R_g^3 / R_c^3,$$

where v represents the part of grain in the whole model volume, equations (12) – (15) become:

$$c_{g1} = \frac{\alpha_g \Delta t}{3} \left[J_g (F - 1) - (G - 1) \frac{HFJ_g + J_c (v - 1)}{v + HG} \right], \tag{16}$$

$$c_{g2} = 0\ , \tag{17}$$

$$c_{c1} = \frac{\alpha_g \Delta t}{3} \left[J_g F - G \frac{HFJ_g + J_c (v - 1)}{v + HG} \right], \tag{18}$$

$$c_{c2} = \frac{\alpha_g \Delta t}{3} \frac{HFJ_g + J_c (v - 1)}{v + HG} R_g^{\ 3}. \tag{19}$$

Let us change now the model described above with a homogeneous sphere made of fictive material with coefficient of the linear thermal expansion α_f. The size of this sphere at the temperature t_0 is the same as the composite sphere, i.e. its radius is R_c. Both spheres are equivalent if their radial displacements are equal at the temperature t

$$\{u_{rg}(R_g) + u_{rc}(R_c)\}_{comp.sphere} = \{u_r(R_c)\}_{fict.sphere}\ ,$$

where $\{u_r(R_c)\}_{fict.sphere} = R_c \alpha_f \Delta t$ follows from the equation similar to the Eq. (3) and boundary conditions $u_{rf}(0) = 0$, $\sigma_{rf}(R_c) = 0$. Then we obtain

$$\alpha_f = \left(c_{g1} + J_g \frac{\alpha_g \Delta t}{3} \right) \frac{R_g}{R_c \Delta t} + c_{c1} \frac{R_c}{R_c \Delta t} + c_{c2} \frac{1}{R_c^{\ 2} R_c \Delta t} + J_c \frac{\alpha_c \Delta t}{3R_c^{\ 2} R_c \Delta t} \left(R_c^{\ 3} - R_g^{\ 3} \right) \tag{20}$$

and after using equations (16) – (19) and after mathematical modifications

$$\alpha_f = \alpha_g \frac{1}{3}\left[F J_g \left(\sqrt[3]{v} + 1 \right) + \frac{HFJ_g + J_c (v-1)}{v + HG} \left(v - G - (G-1)\sqrt[3]{v} \right) \right] + \alpha_c \frac{1}{3}\left[J_c (1 - v) \right]. \qquad (21)$$

All parameters can be expressed using the base material properties as follows

$$F = \frac{E_g (2\mu_c - 1)(5\mu_g - 1)}{(1+\mu_g)[E_g(2\mu_c - 1) + E_c(1 - 2\mu_g)]}, \qquad G = \frac{\dfrac{2E_c}{1+\mu_c} + \dfrac{E_g}{1-2\mu_g}}{\dfrac{E_c}{2\mu_g - 1} + \dfrac{E_g}{1-2\mu_g}},$$

$$H = \frac{1+\mu_c}{2-4\mu_c}, \; J_g = \frac{1+\mu_g}{1-\mu_g}, \; J_c = \frac{1+\mu_c}{1-\mu_c}.$$

Special case when $E_g \approx E_c$, $\mu_g \approx \mu_c$ and $\alpha_g \neq \alpha_c$ have been studied in our former contribution [4].

3 Conclusions

The structure of the two-phase material can be considered as two concentric spheres model. The expansion of such model during its heating can be solved as a thermoelasticity problem. If the material parameters of the inner sphere and outer sphere are constant in the considered temperature region and if their Young's moduli and Poisson's ratios are different, then the coefficient of the linear thermal expansion α_f of the two-phase material can be calculated using the equation (21).

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